

Beyond cash-additive capital requirements: when changing the numeraire fails ¹

WALTER FARKAS², PABLO KOCH-MEDINA³ and COSIMO-ANDREA MUNARI⁴

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Abstract

We discuss general capital requirements representing the minimum amount of capital that a financial institution needs to raise and invest in a pre-specified *eligible*, or *reference*, asset to ensure it is adequately capitalized. Financial positions are modeled as elements belonging to an ordered topological vector space. The payoff of the eligible asset is assumed to be an arbitrary non-zero positive element, thus allowing for a wide range of choices. In the context of function spaces, these general capital requirements cannot be transformed into cash-additive capital requirements by a simple change of numeraire unless the payoff of the eligible asset is bounded away from zero. This excludes the possibility of choosing a defaultable security as the eligible asset which, given the potential unavailability of risk-free assets, constitutes an important gap in the existing theory of capital requirements. This paper fills this gap and provides a detailed analysis of the interplay between acceptance sets and eligible assets. We provide a variety of finiteness and continuity results when the eligible asset has a payoff with “interior-like” qualities, paying particular attention to the case where the underlying space of positions is a Fréchet lattice. As an application we provide a complete characterization of finiteness and L^p -continuity for quantile-based capital requirements, the most important types of capital requirements encountered in practice.

Keywords: ordered topological vector spaces, Fréchet and Banach lattices, acceptance sets, eligible asset, defaultable securities, Value-at-Risk, Tail Value-at-Risk.

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1 Introduction

Since its beginnings, the theory of acceptance sets and capital requirements (risk measures) has had considerable influence on the financial sector as witnessed by the significant impact on modern regulatory solvency regimes, such as the Basel regimes for banks and Solvency II and the Swiss Solvency Test for insurance companies. The theory of coherent acceptance sets and coherent capital requirements was introduced in 1999 in the landmark paper [4] by Artzner, Delbaen, Eber and Heath for finite probability spaces, and in 2000 by Delbaen in [12] for general probability spaces. It was generalized to convex acceptance sets and convex capital requirements by Föllmer and Schied in [19] and, independently, by Frittelli and Rosazza Gianin in [21].

In a one-period setting with dates $t = 0$ and $t = 1$, the space of time $t = 1$ financial positions of a financial institution, i.e. the value of its assets less the value of its liabilities expressed in a fixed numeraire (unit of account), is typically modeled as a Banach lattice \mathcal{X} of real-valued measurable functions on some measurable space (Ω, \mathcal{F}) . A financial institution is deemed adequately capitalized if its position belongs to a pre-specified *acceptance set* $\mathcal{A} \subset \mathcal{X}$. In this context, a *capital requirement* measures the least amount of capital that would need to be raised today, invested in a pre-specified asset — called the *eligible* or *reference* asset— and added to the original position so that the combined position is acceptable.

Distinguishing the eligible asset and the numeraire asset

It is worthwhile being careful about the two assets that play a special role in this measurement procedure: the numeraire and the eligible asset. When cash is used as the numeraire we will speak about *monetary* capital requirements¹. The most widely studied capital requirements in the literature are monetary capital requirements where cash is not only the numeraire, but also the eligible asset. These capital requirements can be formally described as follows, for every position $X \in \mathcal{X}$:

$$\rho_{\mathcal{A}}(X) := \inf \{m \in \mathbb{R}; X + m \in \mathcal{A}\} . \quad (1)$$

Such capital requirements satisfy the following *translation invariance* or *cash-additivity* property for every $X \in \mathcal{X}$ and $m \in \mathbb{R}$

$$\rho_{\mathcal{A}}(X + m) = \rho_{\mathcal{A}}(X) - m , \quad (2)$$

and for this reason are sometimes called *cash-additive* capital requirements.

It is well-known that an alternative interpretation of (1) is possible. Retaining cash as the numeraire but replacing cash as the investment vehicle by a more general eligible asset S with initial price $S_0 > 0$ and terminal payoff $S_1 : \Omega \rightarrow \mathbb{R}_+$, the corresponding capital requirement for $X \in \mathcal{X}$ is given by

$$\rho_{\mathcal{A},S}(X) := \inf \left\{ m \in \mathbb{R}; X + \frac{m}{S_0} S_1 \in \mathcal{A} \right\} . \quad (3)$$

Note that this is still a monetary capital requirement. Capital requirements for eligible assets other than cash are important, also from a practical perspective, because it is in a company's interest to reach

¹Note that there is no consistency in the use of the term *monetary*. Some authors use that term to mean a capital requirement where cash is both the numeraire and the eligible asset, or even, more generally, whenever the same asset is used as both the numeraire and the eligible asset. We believe the present usage reflects better the fact that the capital requirements are expressed in monetary terms.

acceptability at the lowest possible cost, i.e. with the least amount of capital. However, as discussed in Section 2.3 below, restricting the eligible asset to cash will generally be suboptimal since choosing a different eligible asset may lead to a lower capital requirement.

Considering a general eligible asset sometimes leads to a theory which is mathematically equivalent to that of cash-additive capital requirements. Indeed, if the payoff of the eligible asset is *bounded away from zero*, then typically, by choosing S as the new numeraire, the capital requirement $\rho_{\mathcal{A},S}$ can be transformed into a non-monetary capital requirement $\rho_{\mathcal{A}_S}$ defined on discounted positions $\frac{X}{S_1}$ and corresponding to the acceptance set $\mathcal{A}_S := \{\frac{X}{S_1} \in \mathcal{X} ; X \in \mathcal{A}\}$. More precisely, $\rho_{\mathcal{A},S}(X) = S_0 \rho_{\mathcal{A}_S}(\frac{X}{S_1})$ for every position $X \in \mathcal{X}$. Since $\rho_{\mathcal{A}_S}$ satisfies the translation invariance property (2), it is also referred to as a cash-additive capital requirement, even if the additivity property applies in fact to a number of units of the numeraire, and not to a cash amount. Hence, the theory of capital requirements satisfying (2) can be viewed as containing the theory of capital requirements for more general assets as long as their payoff is bounded away from zero.

However, if the payoff of the eligible asset fails to be bounded away from zero, the *change of numeraire* described above is *no longer possible*, either because the payoff can be zero in some state of the world, or because we lose control over the space to which discounted positions belong. As a result, the standard theory of cash-additive capital requirements cannot be applied in this case. But dealing only with eligible assets whose payoff is bounded away from zero is too restrictive. On the one hand, it does not allow for the inclusion of stocks, as typically modeled by lognormal distributions, or of limited liability assets such as options or more realistically modeled stocks. On the other hand, it only allows to consider default-free eligible assets, the existence of which has been strongly questioned in light of the recent financial crisis, and can no longer be taken for granted.

Objective of the paper

The aim of this paper is to fill this gap and investigate capital requirements with respect to a general acceptance set $\mathcal{A} \subset \mathcal{X}$ and a general eligible asset S whose terminal payoff S_1 is only assumed to be a non-negative element in \mathcal{X} . In particular, we allow for the possibility that no default-free assets exist.

Allowing for a defaultable eligible asset seems to us the simplest, most natural, and most operational way to allow for the possibility that a risk-free asset may not exist. In order to, amongst other things, determine capital requirements when no risk-free security exists, El Karoui and Ravanelli in [14] have recently proposed to use cash-subadditive risk measures, i.e. risk measures $\rho : \mathcal{X} \rightarrow \overline{\mathbb{R}}$ satisfying

$$\rho(X + m) \geq \rho(X) - m \quad \text{for all } X \in \mathcal{X}, m > 0.$$

However, as discussed in Remark 2.12 below, if \mathcal{A} is an acceptance set and S is a defaultable security, $\rho_{\mathcal{A},S}$ may or may not be cash-subadditive. As a result, cash-subadditive risk measures do not appear to cover all relevant situations arising from the unavailability of a risk-free security.

In [15] the authors have already considered capital requirements with respect to defaultable securities in spaces of bounded measurable positions. The techniques used in that paper rely heavily on the fact that the interior of the positive cone is nonempty. However, the positive cone of a large number of spaces which are important in applications, such as L^p or Orlicz spaces, does have empty interior.

In this paper, financial positions are assumed to belong to an ordered topological vector space (\mathcal{X}, \leq) with positive cone \mathcal{X}_+ whose interior is possibly empty. This is the natural context for dealing with capital requirements of the form (3). We focus on capital requirements $\rho_{\mathcal{A},S}$, where $\mathcal{A} \subset \mathcal{X}$ is a given acceptance set and the eligible asset S has initial price $S_0 = 1$ and non-zero terminal payoff $S_1 \in \mathcal{X}_+$.

By not restricting our attention to convex or coherent acceptance sets, we are able to cover important examples such as capital requirements based on Value-at-Risk which, in general, fail to be convex. We focus our attention on two properties of $\rho_{\mathcal{A},S}$: finiteness and continuity. Both these properties are important, not only from a mathematical, but also from an economic perspective. In respect of finiteness, if $\rho_{\mathcal{A},S}(X) = \infty$ for a financial position $X \in \mathcal{X}$, then X could not be made acceptable by raising any amount of capital and investing it in the eligible asset, suggesting S is not an *effective* vehicle to reach acceptability. If, on the other hand, $\rho_{\mathcal{A},S}(X) = -\infty$ then we could extract an arbitrary amount of capital by going short on the asset S without compromising the acceptability of X , a situation which is clearly not plausible. In respect of continuity, if $\rho_{\mathcal{A},S}$ fails to be continuous at the position X , then a slight change or misstatement in X might lead to a dramatical change in the corresponding capital requirement: the capital requirement figure will not be *robust* with respect to measurement errors.

Our abstract approach allows to capture the essential aspects — related to the order and topological structure of \mathcal{X} — responsible for the presence, or lack, of finiteness and continuity as a result of the interplay between the acceptance set and the eligible asset. As a consequence, we provide a unifying perspective on the theory of capital requirements in a whole variety of spaces of financial positions commonly encountered in the literature, which complements and extends the standard theory of cash-additive capital requirements by filling the gap between eligible assets whose payoff is bounded away from zero and general, potentially defaultable, eligible assets. Moreover, the set of tools we develop can be fruitfully applied to concrete examples, such as capital requirements based on Value-at-Risk or Tail Value-at-Risk. For these we derive complete characterizations of finiteness and continuity. For instance, we show that capital requirements on L^∞ based on Value-at-Risk are continuous if and only if the asset S is risk-free, i.e. the payoff S_1 is bounded away from zero, while they are never globally continuous on L^p whenever $0 \leq p < \infty$. In particular, lack of continuity at a position X is related to the flatness of the distribution function of X (see Section 5.1). Capital requirements on L^p , for $1 \leq p \leq \infty$, based on Tail Value-at-Risk at the level $0 < \alpha < 1$ are instead always continuous provided that $\mathbb{P}(S_1 < \lambda) < \alpha$ for some $\lambda > 0$ or, equivalently, that $\text{TVaR}_\alpha(S_1) < 0$ (see Section 5.2).

Structure of the paper

The paper is structured as follows. In Section 2 we introduce the basic facts on acceptance sets and capital requirements, and provide references to the existing literature. In Section 3 we discuss finiteness and continuity properties of $\rho_{\mathcal{A},S}$ when the payoff of the eligible asset is by turns an interior point of the positive cone, an order unit, a strictly positive element, or an internal point of the underlying acceptance set. The main results are Theorem 3.13 and Theorem 3.19, where we provide explicit conditions for finiteness and continuity in case of convex and coherent acceptance set, respectively: in particular, finiteness and continuity for coherent capital requirements are shown to be equivalent to S_1 belonging to the core of the acceptance set \mathcal{A} . In Section 4 we assume that the space of financial positions is a Fréchet lattice. Based on Proposition 4.2, showing that the core and the interior of a monotone set in a Fréchet lattice always coincide, we prove a general finiteness criterion for $\rho_{\mathcal{A},S}$ stated in Theorem 4.4. Sufficient conditions for finiteness and continuity in case of convex and coherent acceptance sets in a Fréchet lattice are then shown in Corollary 4.7 and Corollary 4.9. In Section 5 we focus on capital requirements based on Value-at-Risk and Tail-Value-at-Risk acceptability, and we provide a complete characterization of finiteness and continuity on L^p , for $0 \leq p \leq \infty$, when the underlying probability space is nonatomic. We have relegated to the Appendix some useful characterizations of upper and lower semicontinuity for capital requirements.

2 Preliminaries

In this preliminary section we provide definitions and collect some basic properties of acceptance sets, eligible assets and capital requirements in the context of ordered topological vector spaces.

2.1 The space of financial positions and acceptance sets

In the theory of capital requirements, the possible financial positions of a financial institution — net positions of assets and liabilities — are typically assumed to belong to some topological vector space of measurable functions on a given measurable space. In this context, an essential feature of such spaces is that they are ordered topological vector spaces when equipped with the natural (pointwise) ordering.

In this paper, financial positions are assumed to belong to an *ordered (Hausdorff) topological vector space* over \mathbb{R} denoted by \mathcal{X} . We write \mathcal{X}_+ for the *positive cone*. As usual, for $X, Y \in \mathcal{X}$ we will write $X \leq Y$ whenever $Y - X \in \mathcal{X}_+$. Moreover, we will denote by $[Y, Z]$ the *order interval* $\{X \in \mathcal{X} ; Y \leq X \leq Z\}$ for $Y, Z \in \mathcal{X}$. The topological dual of \mathcal{X} is denoted by \mathcal{X}' . The space \mathcal{X}' is itself an ordered vector space when equipped with the pointwise ordering. The corresponding positive cone is the space \mathcal{X}'_+ consisting of all positive continuous linear functionals, i.e. functionals $\psi \in \mathcal{X}'$ such that $\psi(X) \geq 0$ whenever $X \in \mathcal{X}_+$.

If \mathcal{A} is a subset of \mathcal{X} , we denote by $\text{int}(\mathcal{A})$, $\overline{\mathcal{A}}$ and $\partial\mathcal{A}$ the interior, the closure and the boundary of \mathcal{A} , respectively. Moreover, we denote by $\text{core}(\mathcal{A})$ the *core*, or *algebraic interior*, of \mathcal{A} , i.e. the set of all positions $X \in \mathcal{A}$ such that for each $Y \in \mathcal{X}$ there exists $\varepsilon > 0$ with $X + \lambda Y \in \mathcal{A}$ whenever $|\lambda| < \varepsilon$. The elements of $\text{core}(\mathcal{A})$ are called *internal points* of \mathcal{A} . An element $X \notin \text{core}(\mathcal{A}) \cup \text{core}(\mathcal{A}^c)$ is called a *bounding point* of \mathcal{A} . Note that a set $\mathcal{A} \subset \mathcal{X}$ will be called a *cone* if $\lambda\mathcal{A} \subset \mathcal{A}$ for all $\lambda \geq 0$.

We recall the key concept of an acceptance set.

Definition 2.1. A set $\mathcal{A} \subset \mathcal{X}$ is called an *acceptance set* whenever the following two conditions are satisfied:

- (A1) \mathcal{A} is a nonempty, proper subset of \mathcal{X} (non-triviality);
- (A2) if $X \in \mathcal{A}$ and $Y \geq X$ then $Y \in \mathcal{A}$ (monotonicity).

The above definition of an acceptance set encapsulates the minimal properties one would expect from any non-trivial notion of acceptability: some — but not all — positions should be acceptable, and any financial position that dominates, with respect to the chosen ordering, an already accepted position should also be acceptable.

Remark 2.2. (i) Any nonempty intersection of acceptance sets is an acceptance set. If a union of acceptance sets is a strict subset of \mathcal{X} , then it is itself an acceptance set.

(ii) If $\text{int}(\mathcal{A})$ is nonempty, then $\text{int}(\mathcal{A})$ is an acceptance set. Similarly, if $\overline{\mathcal{A}}$ is a strict subset of \mathcal{X} , then $\overline{\mathcal{A}}$ is also an acceptance set.

(iii) If the positive cone \mathcal{X}_+ has nonempty interior, then any acceptance set has nonempty interior. Similarly, if the core of \mathcal{X}_+ is nonempty, then any acceptance set has nonempty core.

(iv) Of particular interest are the classes of *convex* acceptance sets, *conic* acceptance sets (i.e. acceptance sets that are cones), and *coherent* acceptance sets (i.e. acceptance sets that are convex and conic).

We next list the examples of acceptance sets which we will use as the main vehicle to illustrate the application of our results.

Example 2.3. (i) The positive cone \mathcal{X}_+ in any ordered vector space \mathcal{X} is a coherent acceptance set which is sometimes called the worst-case scenario acceptance set.

(ii) (Value-at-Risk acceptance) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and set $L^p := L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ for some $0 \leq p \leq \infty$. For $\alpha \in (0, 1)$ the *Value-at-Risk* of $X \in L^p$ at the level α is defined as

$$\text{VaR}_\alpha(X) := \inf\{m \in \mathbb{R}; \mathbb{P}(X + m < 0) \leq \alpha\} . \quad (4)$$

The set

$$\mathcal{A}_{\alpha,p} := \{X \in L^p; \text{VaR}_\alpha(X) \leq 0\} = \{X \in L^p; \mathbb{P}(X < 0) \leq \alpha\} \quad (5)$$

is a conic acceptance set containing L^p_+ , which is not convex.

(iii) (Tail-Value-at-Risk acceptance) Under the same setting of the previous example, the *Tail Value-at-Risk* of $X \in L^p$ at the level α is defined as

$$\text{TVaR}_\alpha(X) := \frac{1}{\alpha} \int_0^\alpha \text{VaR}_\beta(X) d\beta . \quad (6)$$

Tail Value-at-Risk is also known under the names of *Expected Shortfall*, *Conditional Value-at-Risk*, or *Average Value-at-Risk*. The set

$$\mathcal{A}^{\alpha,p} := \{X \in L^p; \text{TVaR}_\alpha(X) \leq 0\} \quad (7)$$

is a coherent acceptance set for which $L^p_+ \subset \mathcal{A}^{\alpha,p} \subset \mathcal{A}_{\alpha,p}$.

2.2 General capital requirements

We now consider assets S with price $S_0 = 1$ and nonzero payoff $S_1 \in \mathcal{X}_+$ that are traded in a financial market. Note that in the context of function spaces, we are allowing for the possibility that the payoff of S is not bounded away from zero, or is even zero in some future states of the world, so that S can potentially be a defaultable bond, an option or a limited-liability asset. If the financial position of a financial institution is not acceptable with respect to a given acceptance set $\mathcal{A} \subset \mathcal{X}$, it is natural to ask which management actions can turn it into an acceptable position, and at which cost. We allow for one specific management action: raising capital and investing it in a pre-specified traded asset, which we call the *eligible asset*, or the *reference asset*, since it is the only asset that is eligible to modify the risk profile of $X \in \mathcal{X}$.

We will use the standard notation $\overline{\mathbb{R}}$ for the extended real line, i.e. $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$.

Definition 2.4. Let $\mathcal{A} \subset \mathcal{X}$ be an arbitrary set and S a traded asset. The *capital requirement* with respect to \mathcal{A} and S is the function $\rho_{\mathcal{A},S} : \mathcal{X} \rightarrow \overline{\mathbb{R}}$ defined for every $X \in \mathcal{X}$ by

$$\rho_{\mathcal{A},S}(X) := \inf\{m \in \mathbb{R}; X + mS_1 \in \mathcal{A}\} . \quad (8)$$

The asset S will be called the *eligible asset*, or the *reference asset*.

Remark 2.5. Let $\mathcal{A} \subset \mathcal{X}$ be an acceptance set of financial positions, and S a traded asset. Note that, if $\rho_{\mathcal{A},S}(X)$ is a positive number, then we can interpret this number as the “minimum” amount of capital that needs to be raised and invested in the eligible asset to turn the position X into an acceptable position. If it is negative, then it represents the amount of capital that can be extracted from the financial institution without compromising the acceptability of its position. This interpretation needs to be used with caution since it is only true in an approximate sense. In fact, $X + \rho_{\mathcal{A},S}(X)S_1$ does not necessarily belong to \mathcal{A} . If \mathcal{A} is closed, then clearly the infimum in (8) is indeed attained.

Remark 2.6. Consider an acceptance set $\mathcal{A} \subset \mathcal{X}$ and let S be a traded asset. Since \mathcal{A} is by definition a strict nonempty subset of \mathcal{X} , the function $\rho_{\mathcal{A},S}$ cannot be identically equal to either $-\infty$ or ∞ .

Remark 2.7. General capital requirements as defined in Definition 2.4 were essentially introduced by Artzner, Delbaen, Eber and Heath in [4] and later by Jaschke and Küchler in [26], and were then framed in a multi-period setting by Scandolo in [32] and by Frittelli and Scandolo in [22], and, in a fairly abstract mathematical setting, by Hamel in [24] and by Filipović and Kupper in [17]. More recent relevant publications are the paper by Artzner, Delbaen and Koch-Medina [5], by Konstantinides and Kountzakis [27] and by Kountzakis [28]. With the exception of [24] and [28], in the above mentioned papers all relevant results on finiteness and continuity assume, explicitly or implicitly, that the payoff of the eligible asset is an interior point of the positive cone in the underlying space of financial positions. Moreover, none of these papers carried out an investigation of the interplay between acceptance sets and eligible assets, which seems to have first appeared in Farkas, Koch-Medina and Munari [15], in the context of spaces of bounded measurable functions.

Before stating some fundamental properties of capital requirements we introduce some notation and recall some notions related to functions taking values on the extended real line.

Definition 2.8. (i) The *proper domain*, or *domain of finiteness*, of a function $\rho : \mathcal{X} \rightarrow \overline{\mathbb{R}}$ is the set $\text{dom}_{\mathbb{R}}(\rho) := \{X \in \mathcal{X} ; \rho(X) \in \mathbb{R}\}$.
(ii) The *epigraph* of $\rho : \mathcal{X} \rightarrow \overline{\mathbb{R}}$ is defined as $\text{epi}(\rho) := \{(X, \alpha) \in \mathcal{X} \times \mathbb{R} ; \rho(X) \leq \alpha\}$.
(iii) We say that $\rho : \mathcal{X} \rightarrow \overline{\mathbb{R}}$ is *convex*, *subadditive*, or *positively homogeneous*, whenever $\text{epi}(\rho)$ is, respectively, convex, closed under addition, or a cone.

The fundamental properties of $\rho_{\mathcal{A},S}$ are well-understood and are summarized below. They follow immediately from the definition of capital requirements.

Proposition 2.9. *Let $\mathcal{A} \subset \mathcal{X}$ and S be a traded asset. Then $\rho_{\mathcal{A},S} : \mathcal{X} \rightarrow \overline{\mathbb{R}}$ satisfies the following properties (where all equalities and inequalities are in $\overline{\mathbb{R}}$):*

(i) $\rho_{\mathcal{A},S}$ is S -additive, i.e.

$$\rho_{\mathcal{A},S}(X + mS_1) = \rho_{\mathcal{A},S}(X) - m \quad \text{for all } X \in \mathcal{X} \text{ and } m \in \mathbb{R} ; \quad (9)$$

(ii) if \mathcal{A} satisfies the monotonicity axiom (A2), then $\rho_{\mathcal{A},S}$ is monotone (decreasing), i.e.

$$\rho_{\mathcal{A},S}(X) \geq \rho_{\mathcal{A},S}(Y) \quad \text{for all } X, Y \in \mathcal{X} \text{ with } X \leq Y ; \quad (10)$$

(iii) if $\rho : \mathcal{X} \rightarrow \overline{\mathbb{R}}$ is not identically equal to $-\infty$ or to ∞ and it is S -additive and monotone, then $\mathcal{A}_\rho := \{X \in \mathcal{X} ; \rho(X) \leq 0\}$ is an acceptance set and $\rho = \rho_{\mathcal{A}_\rho, S}$;

(iv) if \mathcal{A} is convex, closed under addition, or a cone, then $\rho_{\mathcal{A},S}$ is, respectively, convex, subadditive, or positively homogeneous.

Remark 2.10. Consider an acceptance set $\mathcal{A} \subset \mathcal{X}$ and let S be a traded asset.

(i) It is easy to see that the following inclusions hold:

$$\text{int}(\mathcal{A}) \subset \{X \in \mathcal{X} ; \rho_{\mathcal{A},S}(X) < 0\} \subset \mathcal{A} \subset \{X \in \mathcal{X} ; \rho_{\mathcal{A},S}(X) \leq 0\} \subset \overline{\mathcal{A}} . \quad (11)$$

(ii) Moreover we have $\{X \in \mathcal{X} ; \rho_{\mathcal{A},S}(X) = 0\} \subset \partial\mathcal{A}$. In particular, if $\rho_{\mathcal{A},S}(X)$ is finite for $X \in \mathcal{X}$, then the position $X + \rho_{\mathcal{A},S}(X)S_1$ belongs to the boundary of \mathcal{A} by S -additivity.

Remark 2.11. For notational convenience, we have normalized the price of the eligible asset S to 1, i.e. we have assumed $S_0 = 1$. If we instead consider a price $S_0 > 0$, then the corresponding capital requirement is given for all $X \in \mathcal{X}$ by

$$\rho_{\mathcal{A},S}(X) := \inf \left\{ m \in \mathbb{R} ; X + \frac{m}{S_0} S_1 \in \mathcal{A} \right\} . \quad (12)$$

It is sometimes useful to keep this in mind because it underscores that the capital requirement represents a *capital amount* in monetary terms and not a *number of units* of the eligible asset.

Remark 2.12. As already mentioned, we allow for the possibility that the eligible asset is a defaultable asset. This seems to be the most natural way to deal with the possibility that a risk-free asset may not exist. Moreover, this approach is also very straightforward from an operational perspective which is important in practice. An alternative approach to deal, amongst other things, with the lack of a risk-free asset is using so-called *cash-subadditive* risk measures as recently proposed by El Karoui and Ravanelli in [14] and further pursued by Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio in [8] in the general context of *quasiconvex* risk measures. Cash-subadditive risk measures are risk measures $\rho : \mathcal{X} \rightarrow \overline{\mathbb{R}}$ which instead of satisfying the cash-additivity axiom, satisfy the cash-subadditivity axiom

$$\rho(X + m) \geq \rho(X) - m \quad \text{for all } X \in \mathcal{X}, m > 0 . \quad (13)$$

However, a simple inspection of capital requirements of the form $\rho_{\mathcal{A},S}$ shows that, in general, these risk measures are not cash-subadditive, also in case the asset S is a defaultable bond. Therefore, when no risk-free security exists, demanding cash-subadditivity appears to be too restrictive. In our present framework, the cash-additivity axiom is naturally replaced by the more general *S-additivity* in Proposition 2.9. Depending on S , this general additivity may or may not imply cash-subadditivity.

2.3 Conditions for finiteness and continuity

We end this preliminary section with some general characterizations of when capital requirements are finitely valued and continuous. Some results in this section generalize corresponding results in [15] for bounded measurable financial positions.

Conditions for finiteness

Let $\mathcal{A} \subset \mathcal{X}$ be an acceptance set and S a traded asset. Consider a financial position $X \in \mathcal{X}$. Note that if $\rho_{\mathcal{A},S}(X) = \infty$, then X cannot be made acceptable by raising any amount of capital and investing it in the eligible asset S . This suggests S is not an effective vehicle for changing the risk profile of this position. On the other hand, if $\rho_{\mathcal{A},S}(X) = -\infty$, then we can extract arbitrary amounts of capital retaining the acceptability of X , a property which is clearly not plausible from an economic perspective. It is therefore important to understand when capital requirements are finitely valued.

A general characterization of pointwise and global finiteness for capital requirements is presented in the next proposition. The result easily follows from the definition of capital requirements.

Proposition 2.13. *Let $\mathcal{A} \subset \mathcal{X}$ be an acceptance set and S a traded asset. Then the following statements hold:*

- (i) *for a position $Y \in \mathcal{X}$ we have $\rho_{\mathcal{A},S}(Y) < \infty$ if and only if there exists $m_0 \in \mathbb{R} \cup \{-\infty\}$ such that $Y + mS_1 \in \mathcal{A}$ for any $m > m_0$, and $\rho_{\mathcal{A},S}(Y) > -\infty$ if and only if there exists $m_0 \in \mathbb{R} \cup \{\infty\}$ such that $Y + mS_1 \notin \mathcal{A}$ for any $m < m_0$;*
- (ii) *$\rho_{\mathcal{A},S}(X) < \infty$ for all $X \in \mathcal{X}$ if and only if $\mathcal{A} - \mathbb{R}_+S_1 = \mathcal{X}$ and $\rho_{\mathcal{A},S}(X) > -\infty$ for all $X \in \mathcal{X}$ if and only if $\mathcal{A}^c + \mathbb{R}_+S_1 = \mathcal{X}$.*

Remark 2.14. The results in the above proposition are basic and are also given, in a slightly different form, in [23, Theorem 2.3.1] and in [24, Proposition 9].

If a capital requirement $\rho_{\mathcal{A},S}$ does not attain the value $-\infty$, then, since $\rho_{\mathcal{A},S}$ cannot be identically ∞ , the domain of finiteness is nonempty. The following proposition shows that, in case the acceptance set \mathcal{A} is convex, the core of the domain of finiteness of $\rho_{\mathcal{A},S}$ is not empty if and only if \mathcal{A} has nonempty core. This will turn out to be useful for finiteness and continuity results for capital requirements in the context of Fréchet lattices (see Section 4.2 below).

Proposition 2.15. *Let $\mathcal{A} \subset \mathcal{X}$ be a convex acceptance set and S a traded asset. Assume furthermore that $\rho_{\mathcal{A},S}$ does not attain the value $-\infty$. Then $\text{core}(\text{dom}_{\mathbb{R}}(\rho_{\mathcal{A},S}))$ is nonempty if and only if $\text{core}(\mathcal{A})$ is nonempty.*

Proof. Since $\mathcal{A} \subset \text{dom}_{\mathbb{R}}(\rho_{\mathcal{A},S})$, it is enough to show that if $\text{core}(\text{dom}_{\mathbb{R}}(\rho_{\mathcal{A},S}))$ is nonempty then the same is true for $\text{core}(\mathcal{A})$. Take $X \in \text{core}(\text{dom}_{\mathbb{R}}(\rho_{\mathcal{A},S}))$. It is easy to see, using the additivity of capital requirements, that we may choose X satisfying $\rho_{\mathcal{A},S}(X) < 0$. We show that X belongs to $\text{core}(\mathcal{A})$. Assume $X \notin \text{core}(\mathcal{A})$. Then there exist a non-zero $Y \in \mathcal{X}$ and a sequence (λ_n) of real numbers converging to zero such that $X + \lambda_n Y \notin \mathcal{A}$ for every n . Since $X \in \text{core}(\text{dom}_{\mathbb{R}}(\rho_{\mathcal{A},S}))$, we find $\varepsilon > 0$ such that $X + \lambda Y \in \text{dom}_{\mathbb{R}}(\rho_{\mathcal{A},S})$ for $|\lambda| < \varepsilon$. For $\lambda \in (-\varepsilon, \varepsilon)$ set $f(\lambda) := \rho_{\mathcal{A},S}(X + \lambda Y)$. Note that f is a real-valued convex function defined on the interval $(-\varepsilon, \varepsilon)$ and is therefore continuous. It follows that $\rho_{\mathcal{A},S}(X + \lambda_n Y) \rightarrow \rho_{\mathcal{A},S}(X)$. This implies that $\rho_{\mathcal{A},S}(X + \lambda_n Y) < 0$ for some n , hence $X + \lambda_n Y \in \mathcal{A}$ by monotonicity, contradicting the above assumption that $X + \lambda_n Y \notin \mathcal{A}$ for every n . In conclusion, X must belong to $\text{core}(\mathcal{A})$. \square

As an immediate corollary to the above proposition we state a necessary condition for a convex risk measure to be finitely valued.

Corollary 2.16. *Let $\mathcal{A} \subset \mathcal{X}$ be a convex acceptance set, and S a traded asset. If $\rho_{\mathcal{A},S}$ is finitely valued, then $\text{core}(\mathcal{A})$ is nonempty.*

Remark 2.17. This necessary condition lacks any informational value in spaces for which the positive cone has nonempty core — such as spaces of bounded measurable functions — since in those spaces every acceptance set has nonempty core. However, if we assume that the positive cone has empty core, then capital requirements with respect to the positive cone, or other acceptance sets with empty core, can never be finitely valued, regardless of which eligible asset is chosen (see also Remark 2.23).

Next we provide a useful finiteness criterium for convex risk measures.

Proposition 2.18. *Let $\mathcal{A} \subset \mathcal{X}$ be a convex acceptance set, and S a traded asset. Assume $\rho_{\mathcal{A},S}$ does not attain the value $-\infty$. If \mathcal{A} has nonempty interior and $\rho_{\mathcal{A},S}$ is finitely valued on a dense linear subspace \mathcal{S} of \mathcal{X} , then $\rho_{\mathcal{A},S}$ is finitely valued on the whole \mathcal{X} .*

Proof. Assume $X \notin \text{dom}_{\mathbb{R}}(\rho_{\mathcal{A},S})$. Since the domain of $\rho_{\mathcal{A},S}$ is convex and contains \mathcal{A} , by standard separation we can find a nonzero $\psi \in \mathcal{X}'$ such that $\psi(X) < \psi(\lambda Y)$ for all $\lambda \in \mathbb{R}$ and $Y \in \mathcal{S}$. But this implies ψ must annihilate \mathcal{S} , and hence, by density, the whole space \mathcal{X} . Since this is a contradiction, $\rho_{\mathcal{A},S}$ must be finitely valued on the whole \mathcal{X} . \square

Remark 2.19. The previous result can be easily generalized to the following statement: if \mathcal{X} is a topological vector space, then every convex map $\rho : \mathcal{X} \rightarrow \mathbb{R} \cup \{\infty\}$, whose effective domain $\text{dom}(\rho)$ has nonempty interior and contains a dense linear subspace, is finitely valued.

Conditions for continuity

We present here a useful result providing a pointwise characterization of continuity for the capital requirement $\rho_{\mathcal{A},S}$ without making any assumption on either the acceptance set or the eligible asset. The result is a straightforward consequence of the results on semicontinuity included in the Appendix (see Proposition 6.1 and Proposition 6.2).

Proposition 2.20. *Consider an acceptance set $\mathcal{A} \subset \mathcal{X}$ and a traded asset S .*

- (i) *Assume $X \in \mathcal{X}$. The following statements are equivalent:*
 - (a) $\rho_{\mathcal{A},S}$ is continuous at $X \in \mathcal{X}$;
 - (b) $\rho_{\text{int}(\mathcal{A}),S}(X) = \rho_{\mathcal{A},S}(X) = \rho_{\overline{\mathcal{A}},S}(X)$;
 - (c) $X + mS_1 \notin \overline{\mathcal{A}}$ for $m < \rho_{\mathcal{A},S}(X)$ and $X + mS_1 \in \text{int}(\mathcal{A})$ for $m > \rho_{\mathcal{A},S}(X)$.
- (ii) *The following statements are equivalent:*
 - (a) $\rho_{\mathcal{A},S}$ is (globally) continuous;
 - (b) $\{X \in \mathcal{X} ; \rho_{\mathcal{A},S}(X) \leq 0\}$ is closed and $\{X \in \mathcal{X} ; \rho_{\mathcal{A},S}(X) < 0\}$ is open;
 - (c) $\text{int}(\mathcal{A}) = \{X \in \mathcal{X} ; \rho_{\mathcal{A},S}(X) < 0\}$ and $\overline{\mathcal{A}} = \{X \in \mathcal{X} ; \rho_{\mathcal{A},S}(X) \leq 0\}$.

Remark 2.21. Consider an acceptance set $\mathcal{A} \subset \mathcal{X}$ and a traded asset S .

- (i) Fix a position $X \in \mathcal{X}$ and assume that $\rho_{\mathcal{A},S}(X)$ is finite. Then property (c) in part (i) of Proposition 2.20 can be interpreted as a *transversality condition*: the line $X + mS_1$ comes from outside \mathcal{A} for $m < \rho_{\mathcal{A},S}(X)$, crosses the boundary $\partial\mathcal{A}$ at $m = \rho_{\mathcal{A},S}(X)$, and immediately enters $\text{int}(\mathcal{A})$ for $m > \rho_{\mathcal{A},S}(X)$.
- (ii) If $\rho_{\mathcal{A},S}$ is continuous, then clearly $\partial\mathcal{A} = \{X \in \mathcal{X} ; \rho_{\mathcal{A},S}(X) = 0\}$.

The following necessary condition for a capital requirement to be continuous follows immediately from Proposition 2.20. It is worth stating since it highlights the difference between spaces of bounded measurable functions, or more generally spaces where the positive cone has nonempty interior, and general ordered topological vector spaces.

Corollary 2.22. *Consider an acceptance set $\mathcal{A} \subset \mathcal{X}$ and a traded asset S . If $\rho_{\mathcal{A},S}$ is continuous at some point $X \in \mathcal{X}$ for which $\rho_{\mathcal{A},S}(X) < \infty$, then $\text{int}(\mathcal{A})$ is nonempty. In particular, if $\rho_{\mathcal{A},S}$ is continuous on \mathcal{X} , then $\text{int}(\mathcal{A})$ is nonempty.*

Remark 2.23. Assume $(\Omega, \mathcal{F}, \mathbb{P})$ is a nonatomic probability space and take $1 \leq p < \infty$. The previous corollary, together with Corollary 2.16, shows that Theorem 2.9 in [25] cannot be true in the stated generality, namely that any lower semicontinuous, coherent cash-additive risk measure $\rho : L^p \rightarrow \mathbb{R} \cup \{\infty\}$ must automatically be finitely valued and continuous. In fact, such a risk measure need neither be finite nor continuous. To see this, consider the closed coherent acceptance set $\mathcal{A} := L_+^p$ and the eligible asset S with $S_1 = 1_\Omega$. The corresponding risk measure $\rho_{\mathcal{A},S}$ is coherent, lower semicontinuous, and it is easy to see that it cannot attain the value $-\infty$. Since $\text{core}(L_+^p) = \text{int}(L_+^p) = \emptyset$, we conclude from Corollary 2.16 that $\rho_{\mathcal{A},S}$ cannot be finitely valued. Moreover, by Corollary 2.22 $\rho_{\mathcal{A},S}$ cannot be continuous on any point where it is finitely valued. In fact, the problem with Theorem 2.9 in that paper originates in the proof of [25, Proposition 2.8], where statement (2.12) is proved by deriving pointwise bounds for a linear functional and invoking the uniform boundedness principle. Unfortunately, those pointwise bounds are not necessarily finite.

Although the finiteness of a capital requirement generally does not imply its continuity (see for instance Example 4.4 in [15]), this is true whenever the underlying acceptance set is convex and has nonempty interior. In the spirit of Theorem 1 in [7], the following result can be regarded as an Extended Namioka-Klee theorem for risk measures defined on general ordered topological vector spaces. Note that no lattice structure is required here.

Proposition 2.24. *Let $\mathcal{A} \subset \mathcal{X}$ be a convex acceptance set, and S a traded asset. Assume $\rho_{\mathcal{A},S}$ does not take the value $-\infty$, and $\text{int}(\mathcal{A})$ is nonempty. Then the following hold:*

- (i) $\rho_{\mathcal{A},S}$ is continuous on the interior of its domain;
- (ii) if $\rho_{\mathcal{A},S}$ is finitely valued, then it is continuous on \mathcal{X} ;
- (iii) if \mathcal{X} is an ordered normed space, then $\rho_{\mathcal{A},S}$ is locally Lipschitz continuous on the interior of its domain;
- (iv) if \mathcal{X} is an ordered normed space, \mathcal{A} is coherent, and $\rho_{\mathcal{A},S}$ is finitely valued, then $\rho_{\mathcal{A},S}$ is (globally) Lipschitz continuous on \mathcal{X} .

Proof. First, note that \mathcal{A} is contained in the domain of $\rho_{\mathcal{A},S}$, which has, then, nonempty interior. Since $\rho_{\mathcal{A},S}$ is bounded above by 0 on $\text{int}(\mathcal{A})$, we can apply Theorem 5.43 in [1] to get (i). Assertion (ii) trivially follows from (i).

Assume now that \mathcal{X} is an ordered normed space. Then Theorem 5.44 in [1] yields (iii). Finally, if \mathcal{A} is coherent then, by continuity, $\rho_{\mathcal{A},S}(0) = 0$. Hence, we can find $\alpha > 0$ such that $|\rho_{\mathcal{A},S}(X)| \leq \alpha \|X\|$ for every X in some neighborhood of 0. Since $\rho_{\mathcal{A},S}$ is positively homogeneous, we can extend the previous inequality to all $X \in \mathcal{X}$. Take now arbitrary $X, Y \in \mathcal{X}$. Then, by subadditivity, $\rho_{\mathcal{A},S}(X) - \rho_{\mathcal{A},S}(Y) \leq \rho_{\mathcal{A},S}(X - Y) \leq \alpha \|X - Y\|$. Exchanging the roles of X and Y we obtain global Lipschitz continuity, proving (iv). \square

2.4 Equality between capital requirements

Given two capital requirements $\rho_{\mathcal{A},S}$ and $\rho_{\mathcal{B},R}$, it is natural to ask when they coincide. The following result provides an equivalent condition for the equality of two lower semicontinuous risk measures. For an overview on semicontinuity we refer to the Appendix. Note also that capital requirements based on Value-at-Risk on L^p for $0 \leq p \leq \infty$ and on Tail Value-at-Risk on L^p for $1 \leq p \leq \infty$ are lower semicontinuous by Corollary 5.8 and Corollary 5.15, respectively.

Proposition 2.25. *Consider two acceptance sets $\mathcal{A}, \mathcal{B} \subset \mathcal{X}$ and two traded assets S and R with the same initial price $S_0 = R_0 = 1$. Assume that $\rho_{\mathcal{A},S} : \mathcal{X} \rightarrow \overline{\mathbb{R}}$ and $\rho_{\mathcal{B},R} : \mathcal{X} \rightarrow \overline{\mathbb{R}}$ are both lower semicontinuous. Then $\rho_{\mathcal{A},S} = \rho_{\mathcal{B},R}$ if and only if*

$$\overline{\mathcal{A}} = \overline{\mathcal{B}} \quad \text{and} \quad \overline{\mathcal{A}} = \overline{\mathcal{A}} + \{\lambda(S_1 - R_1); \lambda \in \mathbb{R}\}. \quad (14)$$

Proof. By Proposition 6.1 we may assume without loss of generality that \mathcal{A} and \mathcal{B} are both closed.

To prove necessity assume that $\rho_{\mathcal{A},S} = \rho_{\mathcal{B},R}$. Note that $\mathcal{A} = \mathcal{B}$ as a consequence of Proposition 6.2 in the Appendix. It remains to show that if $X \in \mathcal{A}$ and $\lambda \in \mathbb{R}$, then $X + \lambda(S_1 - R_1) \in \mathcal{A}$. But this also follows from Proposition 6.2, since by additivity

$$\rho_{\mathcal{A},S}(X + \lambda(S_1 - R_1)) = \rho_{\mathcal{A},S}(X - \lambda R_1) - \lambda = \rho_{\mathcal{B},R}(X) + \lambda - \lambda = \rho_{\mathcal{A},S}(X) \leq 0. \quad (15)$$

To prove sufficiency take $X \in \mathcal{X}$. For all $m \in \mathbb{R}$ we have $X + mS_1 = X + mR_1 + m(S_1 - R_1)$. Therefore, $X + mR_1 \in \mathcal{A}$ implies $X + mS_1 \in \mathcal{A}$. It follows that $\rho_{\mathcal{A},S}(X) \leq \rho_{\mathcal{B},R}(X)$. By exchanging the roles of S and R we obtain the reverse inequality, concluding the proof. \square

The next proposition shows an interesting consequence of the above characterization of the equality of capital requirements. For a fixed acceptance set $\mathcal{A} \subset \mathcal{X}$, we cannot find two traded assets S and R such that $\rho_{\mathcal{A},S}(X) \leq \rho_{\mathcal{A},R}(X)$ for all positions $X \in \mathcal{X}$ and such that the inequality is strict for some position. In other words, there exists no *optimal* traded asset S such that, for a different traded asset R , the capital requirements $\rho_{\mathcal{A},S}$ is always lower than $\rho_{\mathcal{A},R}$ without being identical to $\rho_{\mathcal{A},R}$. This implies that if a regulator allows financial institutions to make a position acceptable by raising capital and, irrespective of their individual balance sheets, investing this capital amount in the *same* eligible asset — for instance the risk-free security if it exists — then some institutions may be forced to reach acceptability at a higher cost than would have been possible by choosing an alternative eligible asset.

Proposition 2.26. *Consider an acceptance set $\mathcal{A} \subset \mathcal{X}$ and two traded assets S and R with the same initial price $S_0 = R_0 = 1$. Assume that $\rho_{\mathcal{A},S} : \mathcal{X} \rightarrow \overline{\mathbb{R}}$ and $\rho_{\mathcal{A},R} : \mathcal{X} \rightarrow \overline{\mathbb{R}}$ are lower semicontinuous. If $\rho_{\mathcal{A},S}(X) \leq \rho_{\mathcal{A},R}(X)$ for every position $X \in \mathcal{X}$, then $\rho_{\mathcal{A},S} = \rho_{\mathcal{A},R}$.*

Proof. By Proposition 2.25, it suffices to show that $\overline{\mathcal{A}} = \overline{\mathcal{A}} + \{\lambda(S_1 - R_1); \lambda \in \mathbb{R}\}$ holds. Take $\lambda \in \mathbb{R}$ and $X \in \overline{\mathcal{A}}$, and note that $\rho_{\mathcal{A},R}(X) \leq 0$ by Proposition 6.2 in the Appendix. Since $\rho_{\mathcal{A},R}$ dominates $\rho_{\mathcal{A},S}$, additivity implies $\rho_{\mathcal{A},S}(X + \lambda(S_1 - R_1)) \leq 0$. Hence, again by Proposition 6.2, $X + \lambda(R_1 - S_1)$ belongs to $\overline{\mathcal{A}}$, completing the proof. \square

Remark 2.27. A different non-optimality result is obtained by Filipović in [16], where, in the context L^∞ spaces and convex cash-additive risk measures, the author shows that there exists no optimal numeraire leading to lower capital requirements than any other choice of the numeraire. To prove this, acceptance sets consisting of discounted positions are introduced. However, it is unclear to us how to interpret this result from an economic perspective, since acceptability should not depend on the particular numeraire one has chosen to account in. For a more detailed comment, see [15, Remark 5.7].

3 Interplay between the acceptance set and the eligible asset

In this section we investigate finiteness and continuity properties of capital requirements in general ordered topological vector spaces when the payoff of the eligible asset has some “interior-like” quality. We consider four different combinations of the acceptance set and the payoff of the eligible asset. First, a general acceptance set and a payoff in the interior of the positive cone. Second, a general acceptance set and a payoff which is an order unit. Third, a convex acceptance set and a payoff which is strictly positive. Finally, a coherent acceptance set and a payoff which is an internal point of the acceptance set itself.

The distinction between interior points of the positive cone, order units and strictly positive elements is only meaningful when the positive cone has empty interior since, otherwise, these three concepts coincide. However, when the interior of the positive cone is empty, such as for L^p spaces when the underlying probability space is nonatomic, this is a powerful distinction which helps unveil how the choice of the eligible asset affects the properties of capital requirements.

3.1 Interior points of the positive cone and general acceptance sets

Assume that the positive cone of \mathcal{X} has nonempty interior. If the payoff of the eligible asset is an interior point of the positive cone, then the capital requirement $\rho_{\mathcal{A},S}$ is finitely valued and globally continuous regardless of the underlying acceptance set $\mathcal{A} \subset \mathcal{X}$. This is not surprising since, in case \mathcal{X} is a space of bounded measurable functions, the interior points of the positive cone of \mathcal{X} are exactly the positions which are bounded away from zero, and, for these, finiteness and Lipschitz continuity of the corresponding capital requirements are standard (see also [15]).

Proposition 3.1. *Assume the positive cone \mathcal{X}_+ has nonempty interior. Let $\mathcal{A} \subset \mathcal{X}$ be an arbitrary acceptance set and S a traded asset. If the payoff S_1 is an interior point of \mathcal{X}_+ , then $\rho_{\mathcal{A},S}$ is finitely valued and continuous.*

Proof. Fix $X \in \mathcal{X}$ and take $Y \in \mathcal{A}$ and $Z \in \mathcal{A}^c$. Since S_1 is an interior point of \mathcal{X}_+ , it is easy to see that there exists $m_1 > 0$ such that $Y - X \leq m_1 S_1$. Since the monotonicity of \mathcal{A} implies that $X + m_1 S_1 \in \mathcal{A}$, we infer $\rho_{\mathcal{A},S}(X) < \infty$. On the other hand, we can also find $m_2 > 0$ so that $X - Z \leq m_2 S_1$. Thus, $X - m_2 S_1 \notin \mathcal{A}$, since the monotonicity of \mathcal{A} would otherwise imply that $Z \in \mathcal{A}$. Hence $\rho_{\mathcal{A},S}(X) > -\infty$, showing that $\rho_{\mathcal{A},S}$ is finitely valued.

To prove continuity take $X \in \mathcal{X}$ and assume that $X_n \rightarrow X$. Note that for each $k > 0$ the order interval $\frac{1}{k}[-S_1, S_1]$ is a neighborhood of zero by [1, Theorem 9.40]. Hence for every $k > 0$ there exists $n(k)$ such that $-\frac{1}{k}S_1 \leq X_n - X \leq \frac{1}{k}S_1$ whenever $n > n(k)$. Using the monotonicity and S -additivity of $\rho_{\mathcal{A},S}$, we see that $|\rho_{\mathcal{A},S}(X_n) - \rho_{\mathcal{A},S}(X)| \leq \frac{1}{k}$ for arbitrary $k > 0$ and $n > n(k)$. It follows that $\rho_{\mathcal{A},S}(X_n) \rightarrow \rho_{\mathcal{A},S}(X)$ and, hence, $\rho_{\mathcal{A},S}$ is continuous at X . \square

If \mathcal{X} is an ordered normed space, we can sharpen the previous result and obtain Lipschitz continuity. Before showing this, we provide a simple characterization of the interior points of the positive cone in an ordered normed space.

Lemma 3.2. *Let \mathcal{X} be an ordered normed space. Then $S_1 \in \mathcal{X}_+$ is an interior point of the positive cone if and only if there exists a constant $\lambda > 0$ such that $\frac{X}{\|X\|} \leq \lambda S_1$ for every non-zero $X \in \mathcal{X}$.*

Proof. This follows easily since, by [1, Theorem 9.40], S_1 is an interior point of \mathcal{X}_+ if and only if the order interval $[-S_1, S_1]$ is a neighborhood of zero. \square

Corollary 3.3. *Let \mathcal{X} be an ordered normed space, $\mathcal{A} \subset \mathcal{X}$ and acceptance set, and S a traded asset. If S_1 is an interior point of the positive cone, then $\rho_{\mathcal{A},S}$ is finitely valued and Lipschitz continuous on \mathcal{X} .*

Proof. The risk measure $\rho_{\mathcal{A},S}$ is finitely valued by Proposition 3.1. To prove Lipschitz continuity, take two distinct positions $X, Y \in \mathcal{X}$ and choose $\lambda > 0$ as in Lemma 3.2. Then $Y \leq X + \lambda \|X - Y\| S_1$. By monotonicity and S -additivity, we obtain that $\rho_{\mathcal{A},S}(X) - \rho_{\mathcal{A},S}(Y) \leq \lambda \|X - Y\|$. If we exchange X and Y , we see that $|\rho_{\mathcal{A},S}(X) - \rho_{\mathcal{A},S}(Y)| \leq \lambda \|X - Y\|$, concluding the proof. \square

Remark 3.4. Because of Lemma 3.2, the preceding result implies Lemma 3.5 by Filipović and Kupper in [17] when applied to risk measures on an ordered normed space. It also shows that their lemma is in fact a result in normed spaces for which the positive cone has nonempty interior.

3.2 Order units and general acceptance sets

In this section we still denote by \mathcal{X} a general ordered topological vector space, but we no longer assume that its positive cone has nonempty interior. First we consider eligible assets for which the payoff is an order unit.

Definition 3.5. If \mathcal{X} is an ordered vector space, then $Z \in \mathcal{X}_+$ is called an *order unit* whenever for each $X \in \mathcal{X}$ there exists $\lambda > 0$ such that $X \leq \lambda Z$.

The following remarks state some well-known facts about order units.

- Remark 3.6.* (i) If \mathcal{X} is an ordered vector space, the order units are precisely the internal points of the positive cone, i.e. the elements in $\text{core}(\mathcal{X}_+)$ (see [3, Lemma 1.7]).
- (ii) Let \mathcal{X} be an ordered topological vector space. If \mathcal{X}_+ has nonempty interior, then the notions of interior point of the positive cone and order unit coincide.

- (iii) Let \mathcal{X} be a completely metrizable ordered topological vector space, for instance a Fréchet lattice. Then the notion of interior point of the positive cone and order unit coincide. This follows from a standard Baire category argument.

The proof of finiteness in Proposition 3.1 only used that the payoff of the eligible asset was an order unit. Hence, we have the following result.

Proposition 3.7. *Let $\mathcal{A} \subset \mathcal{X}$ be an arbitrary acceptance set and S a traded asset. If the payoff S_1 is an order unit, then $\rho_{\mathcal{A},S}(X)$ is finite for all $X \in \mathcal{X}$.*

The next example shows that if we choose the positive cone as the acceptance set, the condition that the payoff S_1 of the eligible asset is an order unit is not only sufficient, but also necessary for the risk measure $\rho_{\mathcal{A},S}$ to be everywhere finite.

Example 3.8. If $\mathcal{A} := \mathcal{X}_+$, it is easy to see that the risk measure $\rho_{\mathcal{A},S}$ is everywhere finite if and only if the payoff S_1 of the eligible asset is an order unit. As a result, if we assume that \mathcal{X} is an L^p space, with $1 \leq p < \infty$, or a general Orlicz space on a nonatomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then no capital requirement based on the acceptance set \mathcal{X}_+ can ever be everywhere finite, whatever eligible asset is chosen.

3.3 Strictly positive elements and convex acceptance sets

Requiring, as in Proposition 3.7, that the payoff of the eligible asset is an order unit is quite restrictive. Indeed, by Remark 3.6, this is only possible if the core of the positive cone is nonempty, a condition which is not met by a variety of spaces commonly used in the literature, such as L^p spaces, for $1 \leq p < \infty$, or general Orlicz spaces. To obtain finiteness and continuity results in spaces for which the positive cone has empty core, we relax the requirements on the payoff of the eligible asset and only require that it is strictly positive. However, this less restrictive assumption on the eligible asset comes at a price, since we need to assume that the acceptance set is convex.

Definition 3.9. If \mathcal{X} is an ordered topological vector space, then $Z \in \mathcal{X}_+$ is said to be *strictly positive* whenever $\psi(Z) > 0$ for all non-zero $\psi \in \mathcal{X}'_+$.

Remark 3.10. (i) Using Lemma 5.57 in [1], it is easy to prove that a positive element in an ordered topological vector space is strictly positive if and only if it is not a support point for the positive cone. This characterization is useful for identifying strictly positive elements.

- (ii) If \mathcal{X} is an ordered topological vector space, it is easy to show that any order unit is a strictly positive element. However, the converse is not generally true: for instance 1_Ω is strictly positive in L^p , but L^p_+ has empty core.
- (iii) An ordered topological vector space does not necessarily have strictly positive elements (see Exercise 2.2.10 in [3]). However, any separable Banach lattice admits strictly positive elements (see statement (c) at page 266 in [2]). In this case the set of strictly positive elements is even dense in the positive cone by Theorem 8.43 in [1].
- (iv) Let \mathcal{X} be an ordered topological vector space. If \mathcal{X}_+ has nonempty interior, then the notions of interior point of the positive cone, order unit, and strictly positive element coincide. This follows by a straightforward separation argument and by Lemma 5.57 in [1].

We next characterize strictly positive elements in several important spaces of financial positions.

Example 3.11. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

- (i) (nonempty interior) We have already noted that if \mathcal{X} is the space of real-valued bounded measurable functions on (Ω, \mathcal{F}) , the positive cone has nonempty interior and that $Z \in \text{int}(\mathcal{X}_+)$ if and only if Z is bounded away from zero. In particular we should not confuse strictly positive elements with functions that are everywhere strictly positive. Similarly, the positive cone of L^∞ has nonempty interior and its interior consists of those positive elements which are bounded away from zero almost surely. Again here, strictly positive elements are not to be confused with functions which are strictly positive almost surely.
- (ii) (empty interior, nonempty core) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be nonatomic. If we view L^∞ as the dual space of L^1 and endow it with the w^* -topology $\sigma(L^\infty, L^1)$, then it is not difficult to see that the interior of the positive cone is empty. Note that, as in (i), any positive element in L^∞ which is bounded away from zero almost surely is an order unit. Finally, note that the strictly positive elements are precisely the positions $X \in L^\infty$ such that $X > 0$ almost surely. As a result, in this context there are strictly positive elements that are not order units, even if order units exist.
- (iii) (empty core, but strictly positive elements) If $(\Omega, \mathcal{F}, \mathbb{P})$ is nonatomic, then the positive cone of L^p , for $1 \leq p < \infty$, has empty core. Hence, it does not admit any order unit. However, the positions $X \in L^p$ such that $X > 0$ almost surely correspond to the strictly positive elements. The same is true for Orlicz hearts.

We now prove a simple but important separation property of (not necessarily convex) acceptance sets.

Lemma 3.12. *Let $\mathcal{A} \subset \mathcal{X}$ be an acceptance set and consider a linear functional $\psi : \mathcal{X} \rightarrow \mathbb{R}$. If $\inf_{X \in \mathcal{A}} \psi(X) > -\infty$ then ψ is a positive functional.*

Proof. Let $Z \in \mathcal{X}_+$ be arbitrary and fix $X \in \mathcal{A}$. Then, by monotonicity of \mathcal{A} , we have $X + \lambda Z \in \mathcal{A}$ for all $\lambda \geq 0$. Hence, $\psi(X) + \lambda\psi(Z) \geq \inf_{X \in \mathcal{A}} \psi(X) > -\infty$ for all $\lambda \geq 0$, which can only be true if $\psi(Z) \geq 0$. \square

In Proposition 2.24 we saw that, in case of convex acceptance sets with nonempty interior, finiteness always implies continuity. Here we show that we always have finiteness, and hence continuity, whenever the payoff of the eligible asset is strictly positive.

Theorem 3.13. *Let $\mathcal{A} \subset \mathcal{X}$ be a convex acceptance set with nonempty interior. Assume S is a traded asset whose payoff is strictly positive. Assume furthermore that $\rho_{\mathcal{A}, S}$ does not attain the value $-\infty$. The following statements hold:*

- (i) $\rho_{\mathcal{A}, S}$ is finitely valued and continuous;
- (ii) if \mathcal{X} is an ordered normed space, then $\rho_{\mathcal{A}, S}$ is also locally Lipschitz continuous;
- (iii) if \mathcal{X} is an ordered normed space and \mathcal{A} is coherent, then $\rho_{\mathcal{A}, S}$ is also (globally) Lipschitz continuous.

Proof. By Proposition 2.24 we only need to prove that $\rho_{\mathcal{A}, S}$ is finitely valued. Note that $\text{dom}_{\mathbb{R}}(\rho_{\mathcal{A}, S})$, the domain of finiteness of $\rho_{\mathcal{A}, S}$, is a convex set. Since $\mathcal{A} \subset \text{dom}_{\mathbb{R}}(\rho_{\mathcal{A}, S})$, it has also nonempty interior. Hence, if there exists $X \in \mathcal{X} \setminus \text{dom}_{\mathbb{R}}(\rho_{\mathcal{A}, S})$, by a standard separation argument we can find a non-zero functional $\psi \in \mathcal{X}'$ such that $\psi(X) \leq \psi(Y)$ for any $Y \in \text{dom}_{\mathbb{R}}(\rho_{\mathcal{A}, S})$ and, in particular, for any $Y \in \mathcal{A}$.

By Lemma 3.12, it follows that $\psi \in \mathcal{X}'_+$. Thus, since S_1 is strictly positive, we have $\psi(S_1) > 0$. Take $Z \in \text{dom}_{\mathbb{R}}(\rho_{\mathcal{A},S})$, and note that, by S -additivity, $Z + \lambda S_1 \in \text{dom}_{\mathbb{R}}(\rho_{\mathcal{A},S})$ for every $\lambda < 0$. As a result, $\psi(X) \leq \psi(Z) + \lambda \psi(S_1)$ for every $\lambda < 0$, which is impossible since $\psi(S_1) > 0$. In conclusion, $\text{dom}_{\mathbb{R}}(\rho_{\mathcal{A},S}) = \mathcal{X}$, proving (i). The other assertions now follow from Proposition 2.24. \square

Recall that in L^p spaces strictly positive elements correspond to functions which are strictly positive almost surely. Hence, in these spaces, Theorem 3.13 can be applied to assets whose payoffs are not bounded away from zero. For instance, this is true when the payoff of the eligible asset is lognormally distributed or distributed according to any distribution function F such that $F(0) = 0$ and $F(x) > 0$ for every $x > 0$, like the Levy or the exponential distribution.

As the following two examples show, in Theorem 3.13 we cannot dispense either with the assumption that $\text{int}(\mathcal{A})$ is nonempty or with the assumption that S_1 is strictly positive.

Example 3.14. (i) Let $\mathcal{X} := L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ for some nonatomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and some $1 \leq p < \infty$. Then $\mathcal{A} := \mathcal{X}_+$ is a convex acceptance set and $S_1 := 1_{\Omega}$ is a strictly positive element. Here, \mathcal{A} has empty interior and $\rho_{\mathcal{A},S}(X) = \infty$ for all $X \in \mathcal{X}$ with $\mathbb{P}(X < \lambda) > 0$ for any $\lambda \in \mathbb{R}$.
(ii) Let \mathcal{X} be the space of bounded measurable functions on the measurable space (Ω, \mathcal{F}) . Note that here strictly positive elements and interior points coincide since the positive cone has nonempty interior. Then $\mathcal{A} := \mathcal{X}_+$ is a convex acceptance set with nonempty interior. If we assume that Ω is infinite and that $S_1 \in \mathcal{X}_+ \setminus \{0\}$ is not bounded away from zero, then S_1 is not a strictly positive element and $\rho_{\mathcal{A},S}$ cannot be finitely valued by Example 3.8. For instance, we have $\rho_{\mathcal{A},S}(-1_{\Omega}) = \infty$.

3.4 Internal points of conic and coherent acceptance sets

In this section we focus on conic and coherent acceptance sets and characterize those eligible assets for which the corresponding capital requirements are finitely valued and continuous. We start by showing that for a conic acceptance set $\mathcal{A} \subset \mathcal{X}$ we can sharpen Proposition 2.13 to obtain the following characterization of finiteness.

Proposition 3.15. *Assume $\mathcal{A} \subset \mathcal{X}$ is a conic acceptance set and let S be a traded asset.*

(i) *The following statements are equivalent:*

- (a) $\rho_{\mathcal{A},S}(X) < \infty$ for all $X \in \mathcal{X}$;
- (b) $S_1 \in \text{core}(\mathcal{A})$;
- (c) $\mathcal{A} - S_1$ is absorbing.

(ii) *The following statements are equivalent:*

- (a) $\rho_{\mathcal{A},S}(X) > -\infty$ for all $X \in \mathcal{X}$;
- (b) $-S_1 \in \text{core}(\mathcal{A}^c)$;
- (c) $\mathcal{A}^c + S_1$ is absorbing.

Proof. We only prove part (i). The proof of part (ii) proceeds along similar lines.

If (a) holds but $S_1 \notin \text{core}(\mathcal{A})$, we can find $X \in \mathcal{X}$ such that $S_1 + \lambda_n X \notin \mathcal{A}$ for a suitable sequence (λ_n) of strictly positive numbers converging to zero. Equivalently, $X + \frac{1}{\lambda_n} S_1 \notin \mathcal{A}$ for every n . As a result, $\rho_{\mathcal{A},S}(X) \geq \frac{1}{\lambda_n}$, and therefore $\rho_{\mathcal{A},S}(X) = \infty$, contradicting the assumption. Hence $S_1 \in \text{core}(\mathcal{A})$.

Clearly (b) implies (c). Now assume (c). Then for every $X \in \mathcal{X}$ there exists $\lambda > 0$ such that $X + \lambda S_1 \in \lambda \mathcal{A} = \mathcal{A}$. Hence, $\rho_{\mathcal{A},S}(X) \leq \lambda < \infty$. \square

Remark 3.16. The previous result is similar to [23, Proposition 2.3.7] (see in particular Remark 2.3.8 in that book) and [24, Corollary 6], where the convexity of \mathcal{A} is required instead of its monotonicity. The proof in our situation is simpler. Compare also with Section 2 in [26].

By Proposition 3.15, a necessary condition for the finiteness of a capital requirement with respect to a conic acceptance set $\mathcal{A} \subset \mathcal{X}$ is that the payoff of the eligible asset S belongs to the core of \mathcal{A} .

Corollary 3.17. *Let $\mathcal{A} \subset \mathcal{X}$ be a conic acceptance set and S a traded asset. If $\rho_{\mathcal{A},S}$ is finitely valued, then $S_1 \in \text{core}(\mathcal{A})$. In particular, $\text{core}(\mathcal{A})$ is nonempty.*

For coherent acceptance sets, the above condition is also sufficient.

Theorem 3.18. *Assume $\mathcal{A} \subset \mathcal{X}$ is a coherent acceptance set and let S be a traded asset. Then $\rho_{\mathcal{A},S}$ is finitely valued if and only if $S_1 \in \text{core}(\mathcal{A})$.*

Proof. We only need to prove sufficiency. Assume $S_1 \in \text{core}(\mathcal{A})$. By Proposition 3.15 it is enough to establish that $-S_1 \in \text{core}(\mathcal{A}^c)$. Assume to the contrary that $-S_1 \notin \text{core}(\mathcal{A}^c)$. Then either $-S_1 \in \text{core}(\mathcal{A})$ or $-S_1$ is a bounding point of \mathcal{A} . In both cases it is not difficult to show that any non-trivial convex combination of $-S_1$ and S_1 lies in the core of \mathcal{A} , hence in particular $0 \in \text{core}(\mathcal{A})$. As a result \mathcal{A} is an absorbing cone, i.e. $\mathcal{A} = \mathcal{X}$, which is not possible since \mathcal{A} is a proper subset of \mathcal{X} . \square

As a consequence of the above theorem, we obtain the main result of this section, which characterizes the continuity of capital requirements $\rho_{\mathcal{A},S}$ when $\mathcal{A} \subset \mathcal{X}$ is a coherent acceptance set with nonempty interior. We emphasize that, as every other result in this section, this result holds in the general context of an ordered topological vector space.

Theorem 3.19. *Let $\mathcal{A} \subset \mathcal{X}$ be a coherent acceptance set with nonempty interior and S a traded asset. The following statements are equivalent:*

- (a) $S_1 \in \text{int}(\mathcal{A})$;
- (b) $S_1 \in \text{core}(\mathcal{A})$;
- (c) $\rho_{\mathcal{A},S}$ is finitely valued;
- (d) $\rho_{\mathcal{A},S}$ is continuous.

Proof. Clearly (a) implies (b). As a consequence of Theorem 3.18 and Proposition 2.24 we obtain that (b) implies (c) and that (c) in turn implies (d). To prove that (d) implies (a), assume $\rho_{\mathcal{A},S}$ is continuous and $S_1 \notin \text{int}(\mathcal{A})$. Since $S_1 \in \mathcal{A}$ and $\text{int}(\mathcal{A}) = \{X \in \mathcal{X} ; \rho_{\mathcal{A},S}(X) < 0\}$ by Proposition 2.20, we have $\rho_{\mathcal{A},S}(S_1) = 0$. Then positive homogeneity implies $\rho_{\mathcal{A},S}(S_1 + \lambda S_1) = 0$ for every $\lambda > 0$, hence $\rho_{\text{int}(\mathcal{A}),S}(S_1) = \infty$. Therefore, again by Proposition 2.20, we would also obtain $\rho_{\mathcal{A},S}(S_1) = \infty$, generating a contradiction. In conclusion S_1 must belong to $\text{int}(\mathcal{A})$. \square

4 Capital requirements on Fréchet lattices

Throughout this section we assume, unless otherwise stated, that the space of financial positions \mathcal{X} is a Fréchet lattice, i.e. a locally solid, completely metrizable vector lattice (see [1, Chapter 9]). We denote by $X \vee Y$ and $X \wedge Y$ the supremum, respectively the infimum, of the set $\{X, Y\} \subset \mathcal{X}$. For a position $X \in \mathcal{X}$ we use the notation $X^+ := X \vee 0$ for its positive part, $X^- := (-X) \vee 0$ for its negative part, and $|X| := X \vee (-X)$ for its absolute part.

Local solidity means that there exists a neighborhood base of zero consisting of *solid* neighborhoods \mathcal{U} , i.e. such that if $Y \in \mathcal{U}$ and $|X| \leq |Y|$, then $X \in \mathcal{U}$. By [1, Theorem 5.10], the topology on \mathcal{X} is induced by a complete metric d that is invariant with respect to translations, i.e. such that $d(X + Z, Y + Z) = d(X, Y)$ for all positions $X, Y, Z \in \mathcal{X}$.

- Remark 4.1.* (i) The most common type of Fréchet lattice is a Banach lattice, i.e. an ordered Banach space \mathcal{X} equipped with a lattice structure such that if $|X| \leq |Y|$ for $X, Y \in \mathcal{X}$, then $\|X\| \leq \|Y\|$.
- (ii) The space of real-valued bounded measurable functions on a measurable space (Ω, \mathcal{F}) is a Banach lattice. Also, if $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, all spaces L^p , for $1 \leq p \leq \infty$, as well as Orlicz spaces, are examples of Banach lattices.
- (iii) Assume $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. For any $0 \leq p < 1$, the space L^p is a Fréchet lattice but not a Banach lattice. Note that if the underlying probability space is nonatomic, this space is not locally convex and the only continuous linear functional is the zero functional (see Theorem 13.31 and 13.41 in [1]).

4.1 General acceptance sets

Recall that, by a simple Baire category argument, in complete metric spaces, e.g. in Fréchet lattices, the core and the interior of a closed convex set coincide. In particular, this is the case for the positive cone in a Fréchet lattice (see Theorem 8.43 in [1]). Hence, the results in Section 3.1 imply that, when the payoff of the eligible asset is an order unit, every capital requirement is finitely valued and continuous. For this reason, in this section we do not need to focus on order units.

We start by showing that in a Fréchet lattice the core and the interior of any monotone set coincide. This result is essentially a translation to monotone sets of a similar result for monotone functionals on a Banach lattice obtained by Cheridito and Li in Lemma 4.1 in [10].

Proposition 4.2. *Let \mathcal{X} be a Fréchet lattice and assume that $\mathcal{A} \subset \mathcal{X}$ is monotone. Then $\text{core}(\mathcal{A}) = \text{int}(\mathcal{A})$.*

Proof. Clearly, it is enough to show that $\text{core}(\mathcal{A}) \subset \text{int}(\mathcal{A})$. Take $X \in \text{core}(\mathcal{A})$ and assume that $X \notin \text{int}(\mathcal{A})$. It follows that there exists a sequence (Y_n) converging to 0 such that $X + Y_n \notin \mathcal{A}$. By Theorem 8.41 in [1], we also have $|Y_n| \rightarrow 0$. Without loss of generality we may assume that $d(|Y_n|, 0) \leq 4^{-n}$. Setting $\lambda_n := 2^n$ for $n \in \mathbb{N}$ and recalling that the metric d is translation invariant, we see that the series $\sum_n \lambda_n |Y_n|$ converges to an element $Y \in \mathcal{X}$. Since $X \in \text{core}(\mathcal{A})$, it follows that $X - \delta Y \in \mathcal{A}$ for a suitable $\delta > 0$. For sufficiently large m we have $\lambda_m \geq \delta^{-1}$ and, hence, $X - \delta Y \leq X - \delta \lambda_m |Y_m| \leq X + Y_m$. The monotonicity of \mathcal{A} thus implies $X + Y_m \in \mathcal{A}$, contradicting that $X + Y_n \notin \mathcal{A}$ for all n . In conclusion, X must belong to $\text{int}(\mathcal{A})$. \square

We now provide a characterization of strictly positive elements in a locally convex Fréchet lattice (completeness is in fact not required) which will be needed in the proof of Theorem 4.4. The result extends Theorem 4.85 in [2], stated in the context of a normed vector lattice, in a quite straightforward way. We provide a proof since we were unable to find a reference in this generality. Recall first that if \mathcal{X} is a topological vector lattice, then $Z \in \mathcal{X}_+$ is said to be a *quasi-interior point* if the *principal ideal* $\mathcal{I}_Z := \{X \in \mathcal{X} ; \exists \lambda > 0 : |X| \leq \lambda Z\}$, is dense in \mathcal{X} .

Proposition 4.3. *Let \mathcal{X} be a locally convex Fréchet lattice and assume $Z \in \mathcal{X}_+ \setminus \{0\}$. The following statements are equivalent:*

- (a) *Z is a strictly positive element;*
- (b) *Z is a quasi-interior point;*
- (c) *$X \wedge nZ \rightarrow X$ for any $X \in \mathcal{X}_+$.*

Proof. To prove that (a) implies (b), assume that Z is a strictly positive element. By [1, Theorem 8.54] the principal ideal generated by Z is dense in \mathcal{X} if we endow \mathcal{X} with the weak topology. Since \mathcal{I}_Z is convex and the space \mathcal{X} is locally convex, the ideal \mathcal{I}_Z is also dense with respect to the original topology on \mathcal{X} , as a consequence of [1, Theorem 5.98].

To prove that (b) implies (c), take $X \in \mathcal{X}_+$ and assume \mathcal{U} is a solid neighborhood of zero. We just need to prove that there exists $n_0 \geq 0$ such that $X - (X \wedge nZ) \in \mathcal{U}$ for all $n \geq n_0$. By the density of \mathcal{I}_Z in \mathcal{X} , we find $Y \in \mathcal{I}_Z$ such that $X - Y \in \mathcal{U}$. Setting $W := X \wedge Y^+$, it is easy to show that W belongs to \mathcal{I}_Z . In particular, there exists a positive integer n_0 for which $W \leq n_0 Z$. Moreover, using that $|X - Y| = X \vee Y - X \wedge Y$, we obtain $X - W \leq |X - Y|$. Since $X - Y \in \mathcal{U}$ and \mathcal{U} is solid, it follows that $X - W \in \mathcal{U}$. The solidity of \mathcal{U} implies $X - (X \wedge nZ) \in \mathcal{U}$ for all $n \geq n_0$, since $0 \leq X - (X \wedge nZ) \leq X - (X \wedge n_0 Z) \leq X - W$.

Finally, we show that (c) implies (a). Assume that Z is not a strictly positive element, so that $\psi(Z) = 0$ for some non-zero $\psi \in \mathcal{X}'_+$. As a result $\psi(X \wedge nZ) = 0$ for all $X \in \mathcal{X}_+$ and all positive integers n . By continuity we therefore conclude that ψ annihilates the entire positive cone \mathcal{X}_+ , which cannot be true since ψ was assumed to be non-zero. Thus, Z must be a strictly positive element. \square

Our next result, which is in the spirit of Proposition 6.7 by Shapiro, Dentcheva and Ruszczyński in [33], provides a sufficient condition for a capital requirement on a locally convex Fréchet lattice to be finitely valued, and it is the main result of this section. It shows that if \mathcal{X} is a locally convex Fréchet lattice, then we may dispense in Theorem 3.13 with the assumption that the acceptance set is convex, which was needed there to apply standard separation arguments. Several finiteness results known in the literature follow as special cases. Some examples are [34, Theorem 2.3 (i)] by Svindland and [33, Proposition 6.7] by Shapiro, Dentcheva and Ruszczyński for convex cash-additive risk measures on L^p -spaces, and [10, Theorem 4.6] by Cheridito and Li for convex cash-additive risk measures on Orlicz hearts.

Theorem 4.4. *Let \mathcal{X} be a locally convex Fréchet lattice. Assume that $\mathcal{A} \subset \mathcal{X}$ is an acceptance set with nonempty core and that S is a traded asset whose payoff S_1 is strictly positive. Assume furthermore that $\rho_{\mathcal{A}, S}$ does not attain the value $-\infty$. Then $\rho_{\mathcal{A}, S}$ is finitely valued.*

Proof. By Proposition 4.2, it follows that $\text{int}(\mathcal{A}) \neq \emptyset$. The set $\mathcal{A}_0 := \{X \in \mathcal{X} ; \rho_{\mathcal{A}, S}(X) \leq 0\}$ has nonempty interior since $\mathcal{A} \subset \mathcal{A}_0$. Take $Z \in \text{int}(\mathcal{A}_0)$ and choose a neighborhood of zero \mathcal{U} such that $Z + \mathcal{U} \subset \mathcal{A}_0$. Fix $Y \in \mathcal{X}_+$ and note that $Y = Y \wedge (nS_1) + (Y - nS_1)^+$ for any positive integer n .

Then by Proposition 4.3 we have $(Y - nS_1)^+ \rightarrow 0$, so that $-(Y - mS_1)^+ \in \mathcal{U}$ for a sufficiently large m . Note that $Z - (Y - mS_1)^+ \in \mathcal{A}_0$ and $Z - (Y - mS_1)^+ - mS_1 \leq Z - Y$. Hence, by monotonicity, $\rho_{\mathcal{A},S}(Z - Y) \leq m < \infty$. Now take an arbitrary $X \in \mathcal{X}$. Setting $Y := (Z - X)^+$, it follows that $\rho_{\mathcal{A},S}(X) \leq \rho_{\mathcal{A},S}(Z - Y) < \infty$. Hence $\rho_{\mathcal{A},S}$ is finitely valued. \square

Remark 4.5. Let \mathcal{X} be a locally convex Fréchet lattice. None of the above assumptions on the acceptance set $\mathcal{A} \subset \mathcal{X}$ and on the eligible asset S can be dispensed for the capital requirement $\rho_{\mathcal{A},S}$ to be finitely valued. This follows from the examples provided in Example 3.14, since, by Proposition 4.2, the monotonicity of \mathcal{A} implies that $\text{core}(\mathcal{A}) = \text{int}(\mathcal{A})$.

Remark 4.6. Proposition 6.7 in [33] seems to implicitly assume lower semicontinuity. Indeed, when the authors apply a Baire category argument related to the domain of a cash-additive risk measure $\rho : L^p \rightarrow \mathbb{R} \cup \{\infty\}$, with $1 \leq p < \infty$, it is possible to establish that the interior of the *closure* of the level set $\{X \in L^p; \rho(X) \leq 0\}$ is nonempty. However, unless ρ is lower semicontinuous, it is not clear that the same is true for the level set itself.

4.2 Convex and coherent acceptance sets

We continue to consider eligible assets with strictly positive payoff but now focus on convex and coherent acceptance sets. The following result provides a sufficient condition for a convex capital requirement to be finitely valued and continuous. It follows immediately from Proposition 4.2, Theorem 3.13 and Proposition 2.24.

Corollary 4.7. *Let \mathcal{X} be a Fréchet lattice. Assume that $\mathcal{A} \subset \mathcal{X}$ is a convex acceptance set and that S is a traded asset whose payoff S_1 is a strictly positive element. Assume furthermore that $\rho_{\mathcal{A},S}$ does not attain the value $-\infty$. If $\text{core}(\mathcal{A})$ is nonempty, then $\rho_{\mathcal{A},S}$ is finitely valued and continuous.*

Remark 4.8. The preceding corollary sharpens the finiteness result of Theorem 4.4 in the case of a convex acceptance set in a (not necessarily locally convex) Fréchet lattice.

As an immediate consequence of Theorem 3.19 and Proposition 4.2, we obtain a sufficient condition for the continuity of coherent capital requirements.

Corollary 4.9. *Let $\mathcal{A} \subset \mathcal{X}$ be a coherent acceptance set with nonempty core, and let S be a traded asset. If $S_1 \in \text{core}(\mathcal{A})$ then $\rho_{\mathcal{A},S}$ is finitely valued and continuous.*

The above two results not only provide sufficient conditions for the continuity of convex capital requirements, but also for their finiteness. Hence they can be viewed as a strengthening of the Extended Namioka-Klee theorem proved by Biagini and Frittelli in [7, Theorem 1], in the case of capital requirements with respect to an eligible asset with strictly positive payoff.

Remark 4.10. Our approach can also be used to provide a simpler proof of the Extended Namioka-Klee theorem for general convex capital requirements in a Fréchet lattice. Indeed, if $\mathcal{A} \subset \mathcal{X}$ is a convex acceptance set with nonempty core, and $\rho_{\mathcal{A},S}$ does not attain the value $-\infty$, then Proposition 4.2 implies that $\text{int}(\text{dom}_{\mathbb{R}}(\rho_{\mathcal{A},S}))$ is nonempty. Therefore, since $\rho_{\mathcal{A},S}$ is bounded from above by zero on $\text{int}(\mathcal{A})$, it needs to be continuous on the interior of its domain of finiteness by [1, Theorem 5.43]. Note that [7, Theorem 1] applies to general convex and monotone maps on a Fréchet lattice.

If we had required that the acceptance set in Corollary 4.7 was also closed, then we could have obtained the continuity part by a standard result on the continuity of lower semicontinuous convex functions on a completely metrizable topological vector space.

Proposition 4.11. *Let \mathcal{X} be a completely metrizable ordered topological vector space. Assume that $\mathcal{A} \subset \mathcal{X}$ is a closed convex acceptance set with nonempty core, and that S is a traded asset. Assume furthermore that $\rho_{\mathcal{A},S}$ does not attain the value $-\infty$. Then $\text{int}(\text{dom}_{\mathbb{R}}(\rho_{\mathcal{A},S}))$ is nonempty, and $\rho_{\mathcal{A},S}$ is continuous on $\text{int}(\text{dom}_{\mathbb{R}}(\rho_{\mathcal{A},S}))$.*

Proof. Since \mathcal{A} is convex and closed and \mathcal{X} is a complete metric space, the core of \mathcal{A} coincides with the interior of \mathcal{A} . Hence clearly $\text{int}(\text{dom}_{\mathbb{R}}(\rho_{\mathcal{A},S}))$ is nonempty. The statement then follows from [13, Corollary 2.5]. \square

Fréchet lattices with trivial topological dual

We conclude Section 5 by focusing on Fréchet lattices whose topological dual is trivial. A typical example is $L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ for $0 \leq p < 1$, whenever $(\Omega, \mathcal{F}, \mathbb{P})$ is a nonatomic probability space. In spaces where the topological dual is trivial, the only open, convex subsets are the empty set and the whole \mathcal{X} . As a consequence, Corollary 4.7 is not applicable to such spaces. In fact, in the next result we prove that no convex risk measure can be finitely valued or (globally) continuous on a Fréchet lattice with trivial dual.

Proposition 4.12. *Let \mathcal{X} be a Fréchet lattice with trivial dual $\mathcal{X}' = \{0\}$. If $\mathcal{A} \subset \mathcal{X}$ is a convex acceptance set and S an arbitrary traded asset, then $\rho_{\mathcal{A},S}$ is neither finitely valued nor continuous on the whole \mathcal{X} .*

Proof. Since the only nonempty subset of \mathcal{X} that is open and convex is \mathcal{X} itself, then $\text{int}(\mathcal{A})$ must be empty. By Proposition 4.2 and Corollary 2.16, it follows that $\rho_{\mathcal{A},S}$ cannot be finitely valued. Moreover, by Corollary 2.22, it cannot be (globally) continuous. \square

Remark 4.13. Note that Corollary 3 in [7] and the remark thereafter (on L^p spaces with $0 \leq p < 1$) only hold for maps whose domain has nonempty interior. Indeed, consider $L^p := L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ where $0 \leq p < 1$ and $(\Omega, \mathcal{F}, \mathbb{P})$ is nonatomic. Then the dual of L^p is trivial. The positive cone $\mathcal{A} := L^p_+$ is a convex acceptance set. Therefore, taking $S_1 = 1_\Omega$, the capital requirement $\rho_{\mathcal{A},S}$ is convex and non-constant (and is easily seen never to attain the value $-\infty$).

5 Quantile-based capital requirements

In this final section we investigate the effectiveness and robustness properties of capital requirements based on the Value-at-Risk and Tail-Value-at-Risk acceptance sets introduced in Example 2.3. We characterize those eligible assets with non-negative payoff for which the corresponding capital requirements are finitely valued and continuous. We assume throughout this section that $(\Omega, \mathcal{F}, \mathbb{P})$ is a nonatomic probability space and consider the space $L^p := L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ for $0 \leq p \leq \infty$. The case of a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ will not be treated here, but it leads to similar results.

Recall that if $\alpha \in (0, 1)$, then an α -quantile of $X \in L^0$ is any number $q \in \mathbb{R}$ with the property $\mathbb{P}(X < q) \leq \alpha \leq \mathbb{P}(X \leq q)$. The set of all α -quantiles of X is the closed interval $[q_\alpha^-(X), q_\alpha^+(X)]$ where the upper α -quantile is given by

$$q_\alpha^+(X) := \inf\{m \in \mathbb{R}; \mathbb{P}(X \leq m) > \alpha\} = \sup\{m \in \mathbb{R}; \mathbb{P}(X < m) \leq \alpha\}, \quad (16)$$

and the lower α -quantile by

$$q_\alpha^-(X) := \sup\{m \in \mathbb{R}; \mathbb{P}(X < m) < \alpha\} = \inf\{m \in \mathbb{R}; \mathbb{P}(X \leq m) \geq \alpha\}. \quad (17)$$

Recall from Example 2.3 that

$$\text{VaR}_\alpha(X) = \inf\{m \in \mathbb{R}; \mathbb{P}(X + m < 0) \leq \alpha\} = -q_\alpha^+(X), \quad (18)$$

and

$$\text{TVaR}_\alpha(X) = \frac{1}{\alpha} \int_0^\alpha \text{VaR}_\beta(X) d\beta. \quad (19)$$

Remark 5.1. (i) From [20, Section 4.4] we know that, as a function on L^0 , VaR_α is finitely valued, cash-additive, monotone, and positively homogeneous. Moreover, by [20, Lemma 4.3], VaR_α is globally Lipschitz as a function on L^∞ .

(ii) From [20, Section 4.4] we know that, as a function on L^1 , TVaR_α is finitely valued, cash-additive, monotone, convex. Moreover it follows from [20, Lemma 4.46] that TVaR_α is globally Lipschitz continuous on L^1 . Indeed, one could easily adapt [29, Proposition 2.35], which is given under a slightly different definition of Tail Value-at-Risk (namely in terms of lower quantiles instead of upper quantiles).

(iii) It is possible to also consider TVaR_α on L^p for $0 \leq p < 1$. However, in this case Proposition 4.12 would imply that capital requirements based on this extended range could never be either finitely valued or continuous. For this reason we restrict our attention to the case $1 \leq p \leq \infty$.

In Example 2.3 we had introduced for each $\alpha \in (0, 1)$ the VaR_α -acceptance set in L^p , for $0 \leq p \leq \infty$,

$$\mathcal{A}_{\alpha,p} := \{X \in L^p; \text{VaR}_\alpha(X) \leq 0\}, \quad (20)$$

and the TVaR_α -acceptance set in L^p , for $0 \leq p \leq \infty$ (but we will always assume $p \geq 1$ by the previous remark),

$$\mathcal{A}^{\alpha,p} := \{X \in L^p; \text{TVaR}_\alpha(X) \leq 0\}. \quad (21)$$

5.1 Capital requirements based on Value-at-Risk

We start by describing the core of the acceptance set $\mathcal{A}_{\alpha,p}$. This will help characterize when capital requirements based on $\mathcal{A}_{\alpha,p}$ are finitely valued.

Proposition 5.2. *Let $\alpha \in (0, 1)$. Then the following statements hold:*

- (i) *if $0 \leq p < \infty$ then $\text{int}(\mathcal{A}_{\alpha,p}) = \text{core}(\mathcal{A}_{\alpha,p}) = \{X \in L^p; \mathbb{P}(X \leq 0) < \alpha\}$;*
- (ii) *$\text{int}(\mathcal{A}_{\alpha,\infty}) = \text{core}(\mathcal{A}_{\alpha,\infty}) = \{X \in L^\infty; \text{VaR}_\alpha(X) < 0\}$.*

Proof. By Proposition 4.2 we have $\text{int}(\mathcal{A}_{\alpha,p}) = \text{core}(\mathcal{A}_{\alpha,p})$ for every $0 \leq p \leq \infty$.

(i) Take $X \in L^p$ and assume $\mathbb{P}(X \leq 0) < \alpha$. If $X \notin \text{core}(\mathcal{A}_{\alpha,p})$, then we can find $Z \in L_+^p$ and $\lambda_n \downarrow 0$ such that $\mathbb{P}(X < \lambda_n Z) > \alpha$. As a result $\mathbb{P}(X \leq 0) \geq \mathbb{P}(\bigcap_n \{X < \lambda_n Z\}) \geq \alpha$, contradicting the assumption. Hence $\{X \in L^p; \mathbb{P}(X \leq 0) < \alpha\} \subset \text{core}(\mathcal{A}_{\alpha,p})$.

Now take $X \in \text{core}(\mathcal{A}_{\alpha,p})$ and assume $\mathbb{P}(X \leq 0) \geq \alpha$. Note that we necessarily have $\mathbb{P}(X > 0) > 0$ and, thus, $\mathbb{P}(0 < X < \varepsilon) > 0$ for some $\varepsilon > 0$. Since $(\Omega, \mathcal{F}, \mathbb{P})$ is nonatomic we can find a sequence (A_n) of disjoint subsets of $\{0 < X < \varepsilon\}$ with $0 < \mathbb{P}(A_n) < n^{-p-2}$. Consider $Z := 1_{\{X \leq 0\}} + \sum_n n 1_{A_n} \in L_+^p$. Then for every $\lambda > 0$ there exists n such that $\mathbb{P}(X < \lambda Z) \geq \mathbb{P}(X \leq 0) + \mathbb{P}(A_n) > \alpha$. But this contradicts the assumption that $X \in \text{core}(\mathcal{A}_{\alpha,p})$. In conclusion, $\text{core}(\mathcal{A}_{\alpha,p}) \subset \{X \in L^p; \mathbb{P}(X \leq 0) < \alpha\}$, proving (i).

(ii) This follows immediately from the continuity of VaR_α on L^∞ . \square

Effectiveness

Since $\mathcal{A}_{\alpha,p}$ is a cone, we can provide a characterization of the effectiveness of the capital requirement $\rho_{\mathcal{A}_{\alpha,p},S}$ based on Proposition 3.15.

Proposition 5.3. *Assume $0 \leq p < \infty$. Let $\alpha \in (0, 1)$ and let S be a traded asset. Then the following statements hold:*

- (i) $\rho_{\mathcal{A}_{\alpha,p},S}(X) < \infty$ for all $X \in L^p$ if and only if $\mathbb{P}(S_1 = 0) < \alpha$;
- (ii) $\rho_{\mathcal{A}_{\alpha,p},S}(X) > -\infty$ for all $X \in L^p$ if and only if $\mathbb{P}(S_1 = 0) < 1 - \alpha$.

In particular, $\rho_{\mathcal{A}_{\alpha,p},S}$ is finitely valued if and only if $\mathbb{P}(S_1 = 0) < \min\{\alpha, 1 - \alpha\}$.

Proof. (i) By Proposition 3.15 and Proposition 5.2 it follows directly that (i) holds.

(ii) If $\mathbb{P}(S_1 = 0) \geq 1 - \alpha$, then it is easy to show that $\rho_{\mathcal{A}_{\alpha,p},S}(0) = -\infty$. Now assume $\mathbb{P}(S_1 = 0) < 1 - \alpha$ and take $X \in L^p$. Since $\mathbb{P}(\{X < nS_1\} \cap \{S_1 > 0\}) \uparrow \mathbb{P}(S_1 > 0)$ and $\mathbb{P}(S_1 > 0) > \alpha$, we have that $\mathbb{P}(X - nS_1 < 0) > \alpha$ for n sufficiently large. As a result, $\rho_{\mathcal{A}_{\alpha,p},S}(X) > -\infty$. \square

Proposition 5.4. *Let $\alpha \in (0, 1)$ and let S be a traded asset. Then the following statements hold:*

- (i) $\rho_{\mathcal{A}_{\alpha,\infty},S}(X) < \infty$ for all $X \in L^\infty$ if and only if $\text{VaR}_\alpha(S_1) < 0$;
- (ii) $\rho_{\mathcal{A}_{\alpha,\infty},S}(X) > -\infty$ for all $X \in L^\infty$ if and only if $\text{VaR}_\alpha(-S_1) > 0$.

In particular, $\rho_{\mathcal{A}_{\alpha,\infty},S}$ is finitely valued if and only if $\text{VaR}_\alpha(S_1) < 0 < \text{VaR}_\alpha(-S_1)$.

Proof. (i) By Proposition 3.15 and Proposition 5.2 it follows directly that (i) holds.

(ii) Assume that $\text{VaR}_\alpha(-S_1) > 0$. Then $\mathbb{P}(S_1 > \lambda) > \alpha$ for some $\lambda > 0$. Since for every $X \in L^\infty$ we have $\{S_1 > \lambda\} \subset \{X < \lambda^{-1} \|X\|_\infty S_1\}$, it follows that $\rho_{\mathcal{A}_{\alpha,\infty},S}$ never attains the value $-\infty$. On the other hand, if $\text{VaR}_\alpha(-S_1) \leq 0$ then it is easy to show that $\rho_{\mathcal{A}_{\alpha,\infty},S}(1_\Omega) = -\infty$. \square

Remark 5.5. 1. Consider a traded asset S and let $\alpha \in (0, 1)$. As an immediate consequence of Proposition 5.3 and Proposition 5.4, if $S_1 > 0$ almost surely, then $\rho_{\mathcal{A}_{\alpha,p},S}$ is finitely valued on L^p for every $0 \leq p \leq \infty$.

2. Consider a traded asset S and let $0 < \alpha < \frac{1}{2}$. Then $\rho_{\mathcal{A}_{\alpha,\infty},S}$ is finitely valued if and only if $\text{VaR}_\alpha(S_1) < 0$. Indeed if $\alpha < \frac{1}{2}$ then it is not difficult to show that $\text{VaR}_\alpha(X) + \text{VaR}_\alpha(-X) \geq 0$ for all $X \in L^\infty$. The statement now follows by applying Proposition 5.4.

Robustness

We show that the acceptance set $\mathcal{A}_{\alpha,p}$ is closed in L^p for every $0 \leq p \leq \infty$. This will immediately imply that the capital requirement $\rho_{\mathcal{A}_{\alpha,p},S}$ is lower semicontinuous on L^p for every choice of the traded asset S .

Proposition 5.6. *Let $\alpha \in (0,1)$. Then the acceptance set $\mathcal{A}_{\alpha,p}$ is closed in L^p for every $0 \leq p \leq \infty$.*

Proof. Fix $\alpha \in (0,1)$. It is enough to show that $\mathcal{A}_{\alpha,0}$ is closed in L^0 . Take a sequence (X_n) in $\mathcal{A}_{\alpha,0}$ converging in probability to $X \in L^0$. Fix an arbitrary $m > 0$ and take $0 < \varepsilon < m$. Then set $A_n := \{|X_n - X| > \varepsilon\}$. We easily obtain that $\mathbb{P}(X + m < 0) \leq \mathbb{P}(A_n) + \mathbb{P}(X_n + m - \varepsilon < 0) \leq \mathbb{P}(A_n) + \alpha$ for every n . Thus, by convergence in probability, we conclude that $\mathbb{P}(X + m < 0) \leq \alpha$ for all $m > 0$, i.e. $X \in \mathcal{A}_{\alpha,0}$. As a result, $\mathcal{A}_{\alpha,0}$ is closed in L^0 . \square

Remark 5.7. Note that, by Remark 2.5, the closedness of $\mathcal{A}_{\alpha,p}$ for $0 \leq p \leq \infty$ implies that, in the context of L^p -spaces, in the definition (18) of VaR_α we may replace the infimum by a minimum, i.e.

$$\text{VaR}_\alpha(X) = \min\{m \in \mathbb{R}; \mathbb{P}(X + m < 0) \leq \alpha\}. \quad (22)$$

Combining Remark 6.3 (i) and Proposition 5.6 we obtain the following result.

Corollary 5.8. *Assume $\alpha \in (0,1)$ and let S be an arbitrary traded asset. Then $\rho_{\mathcal{A}_{\alpha,p},S}$ is lower semicontinuous on L^p for every $0 \leq p \leq \infty$.*

The next result characterizes continuity for VaR_α -based capital requirements on L^∞ . The proof is identical to that of Proposition 4.12 in [15] for bounded measurable positions.

Proposition 5.9. *Assume $\alpha \in (0,1)$ and let S be a traded asset. Then $\rho_{\mathcal{A}_{\alpha,\infty},S}$ is (globally) continuous on L^∞ if and only if the payoff S_1 is (essentially) bounded away from zero.*

The next proposition shows that, when $p < \infty$, the capital requirement $\rho_{\mathcal{A}_{\alpha,p},S}$ is never continuous on L^p , regardless of the choice of the traded asset S .

Proposition 5.10. *Assume $0 \leq p < \infty$ and $\alpha \in (0,1)$. Let S be an arbitrary traded asset. Then $\rho_{\mathcal{A}_{\alpha,p},S}$ is not (globally) continuous on L^p .*

Proof. Take $\varepsilon > 0$ and $A \in \mathcal{F}$ with $\mathbb{P}(A) = \alpha$. Then set $X := -(S_1 + \varepsilon)1_A \in L^p$. Using Proposition 5.2 it is easy to show that $\rho_{\text{int}(\mathcal{A}_{\alpha,p}),S}(X) \geq 1$, while $\rho_{\mathcal{A}_{\alpha,p},S}(X) \leq 0$. Hence, by the pointwise characterization of continuity obtained in Proposition 2.20, it follows that $\rho_{\mathcal{A}_{\alpha,p},S}$ is not continuous at X . \square

Using the pointwise characterization of continuity in Proposition 2.20, it is possible to characterize the points of continuity of $\rho_{\mathcal{A}_{\alpha,p},S}$ when $p < \infty$. This characterization takes a particularly nice form in case $S_1 = 1_\Omega$, i.e. for the capital requirement VaR_α . Indeed, the lack of continuity of VaR_α at a position $X \in L^p$ is shown to be related to the flatness of the cumulative distribution function of X at the level α .

Proposition 5.11. *Assume $0 \leq p < \infty$, and let $\alpha \in (0,1)$. For every $X \in L^p$ the following two statements are equivalent:*

- (a) VaR_α is continuous at X ;
- (b) $q_\alpha^-(X) = q_\alpha^+(X)$.

Proof. Following the proof of Proposition 5.6 it is not difficult to show that the lower α -quantile q_α^- is lower semicontinuous on L^p . Now consider the acceptance set $\mathcal{B} := \{X \in L^p; \mathbb{P}(X < 0) < \alpha\}$, and note that, choosing $S_1 := 1_\Omega$, we have $\rho_{\mathcal{B},S} = -q_\alpha^-$. As a result, the risk measure $\rho_{\mathcal{B},S}$ is upper semicontinuous on the whole L^p . Moreover $\text{int}(\mathcal{A}_{\alpha,p}) \subset \mathcal{B} \subset \mathcal{A}_{\alpha,p}$ by Proposition 5.2, hence $\text{int}(\mathcal{A}_{\alpha,p}) = \text{int}(\mathcal{B})$. Then Proposition 6.1 implies that $-q_\alpha^- = \rho_{\mathcal{B},S} = \rho_{\text{int}(\mathcal{B}),S} = \rho_{\text{int}(\mathcal{A}_{\alpha,p}),S}$. Furthermore we have $-q_\alpha^+ = \text{VaR}_\alpha = \rho_{\mathcal{A}_{\alpha,p},S}$. Since VaR_α is lower semicontinuous by Corollary 5.8, the statement easily follows from Proposition 2.20. \square

Remark 5.12. Assume $0 \leq p < \infty$ and take $X \in L^p$. It is well known that $q_\alpha^-(X) = q_\alpha^+(X)$ except for at most countably many values of $\alpha \in (0, 1)$. Hence, VaR_α is continuous at X except for at most countably many values of $\alpha \in (0, 1)$.

As an immediate corollary, we obtain the following result expressed in terms of cumulative distribution functions.

Corollary 5.13. *Assume $0 \leq p < \infty$, and fix $X \in L^p$. Then VaR_α is continuous at X for every $\alpha \in (0, 1)$ if and only if the cumulative distribution function F_X of X is strictly increasing on the interval $I := \{m \in \mathbb{R}; 0 < F_X(m) < 1\}$.*

5.2 Capital requirements based on Tail Value-at-Risk

First we establish the closedness of the acceptance set $\mathcal{A}^{\alpha,p}$, for $1 \leq p \leq \infty$, and give a description of its core. Both statements follow directly from the continuity of TVaR_α on L^1 .

Proposition 5.14. *Assume $1 \leq p \leq \infty$ and $\alpha \in (0, 1)$. Then $\text{core}(\mathcal{A}^{\alpha,p}) = \text{int}(\mathcal{A}^{\alpha,p}) = \{X \in L^p; \text{TVaR}_\alpha(X) < 0\}$. Moreover, $\mathcal{A}^{\alpha,p}$ is closed in L^p .*

Combining Remark 6.3 (i) and Proposition 5.14 we immediately obtain the following corollary.

Corollary 5.15. *Assume $1 \leq p \leq \infty$ and $\alpha \in (0, 1)$. Let S be an arbitrary traded asset. Then $\rho_{\mathcal{A}^{\alpha,p},S}$ is lower semicontinuous on L^p .*

The following proposition characterizes the finiteness and the continuity of $\mathcal{A}^{\alpha,p}$ -based capital requirements.

Proposition 5.16. *Assume $1 \leq p \leq \infty$ and $\alpha \in (0, 1)$. Let S be a traded asset. Then the following statements are equivalent:*

- (a) $\rho_{\mathcal{A}^{\alpha,p},S}$ is finitely valued;
- (b) $\rho_{\mathcal{A}^{\alpha,p},S}$ is (globally) Lipschitz continuous;
- (c) $\text{TVaR}_\alpha(S_1) < 0$;
- (d) there exists $\lambda > 0$ such that $\mathbb{P}(S_1 < \lambda) < \alpha$.

Proof. It follows immediately from Theorem 3.19 that (a) and (c) are equivalent to the continuity of $\rho_{\mathcal{A}^{\alpha,p},S}$. Moreover, they are also equivalent to Lipschitz continuity by Proposition 2.24. To prove that (c) implies (d), note that if $\text{TVaR}_\alpha(S_1) < 0$ there must exist $0 < \beta < \alpha$ such that $\text{VaR}_\beta(S_1) < 0$, since $S_1 \geq 0$. Then $\mathbb{P}(S_1 < \lambda) \leq \beta < \alpha$ for some $\lambda > 0$. Conversely, assume (d) and let $\text{TVaR}_\alpha(S_1) = 0$. Then $\text{VaR}_\beta(S_1) = 0$ for every $0 < \beta < \alpha$. As a consequence, for arbitrary $\lambda > 0$, we have $\mathbb{P}(S_1 < \lambda) > \beta$ for any $0 < \beta < \alpha$, hence $\mathbb{P}(S_1 < \lambda) \geq \alpha$, in contrast to (d). Therefore we must have $\text{TVaR}_\alpha(S_1) < 0$, completing the proof. \square

Remark 5.17. (i) Note that whenever the payoff S_1 of the eligible asset is a strictly positive element in L^p , with $p \geq 1$, i.e. whenever S_1 is strictly positive almost surely, then $\text{TVaR}_\alpha(S_1) < 0$, hence the capital requirement $\rho_{\mathcal{A}^{\alpha,p},S}$ is finitely valued and Lipschitz continuous on L^p .

(ii) In contrast to the case of Value-at-Risk acceptability on L^p , capital requirements based on Tail Value-at-Risk may be finitely valued and continuous even if the payoff of the eligible asset is not bounded away from zero, and even if it is not everywhere positive. It has been often argued that acceptability based on Tail-Value-at-Risk is preferable to acceptability based on Value-at-Risk since in contrast to the former, the latter cannot detect tail-behavior. Given the importance of continuity from an operational perspective, the better continuity and finiteness properties of Tail-Value-at-Risk can be seen as an additional reason to prefer basing acceptability on Tail-Value-at-Risk rather than on Value-at-Risk.

6 Appendix: semicontinuity of capital requirements

This appendix provides various characterizations of pointwise and global upper and lower semicontinuity of capital requirements on general ordered topological vector spaces. The material presented here is essentially an extension of results for bounded measurable financial positions in [15]; compare also Corollary 8 and Corollary 9 by Hamel in [24].

Recall that a function $\rho : \mathcal{X} \rightarrow \overline{\mathbb{R}}$ is said to be *lower semicontinuous at a point* $X \in \mathcal{X}$ if for every $\varepsilon > 0$ there exists a neighborhood \mathcal{U} of X such that $\rho(Y) \geq \rho(X) - \varepsilon$ for all $Y \in \mathcal{U}$. We say that ρ is (globally) *lower semicontinuous* if it is lower semicontinuous at each point $X \in \mathcal{X}$. Note that ρ is lower semicontinuous if and only if $\text{epi}(\rho)$ is closed, or, equivalently, if the set $\{X \in \mathcal{X} ; \rho(X) \leq \lambda\}$ is closed for every $\lambda \in \mathbb{R}$. The function ρ is *upper semicontinuous at a point* $X \in \mathcal{X}$ if $-\rho$ is lower semicontinuous at X and (globally) upper semicontinuous if $-\rho$ is lower semicontinuous. Finally, note that $\rho : \mathcal{X} \rightarrow \overline{\mathbb{R}}$ is continuous at $X \in \mathcal{X}$ if and only if it is both lower and upper semicontinuous at X .

Proposition 6.1 (Pointwise semicontinuity). *Consider an acceptance set $\mathcal{A} \subset \mathcal{X}$ and a traded asset S and assume $X \in \mathcal{X}$.*

(i) *The following statements are equivalent:*

- (a) $\rho_{\mathcal{A},S}$ is lower semicontinuous at X ;
- (b) $X + mS_1 \notin \overline{\mathcal{A}}$ for any $m < \rho_{\mathcal{A},S}(X)$;
- (c) $\rho_{\overline{\mathcal{A}},S}(X) = \rho_{\mathcal{A},S}(X)$.

(ii) *The following statements are equivalent:*

- (a) $\rho_{\mathcal{A},S}$ is upper semicontinuous at X ;
- (b) $X + mS_1 \in \text{int}(\mathcal{A})$ for any $m > \rho_{\mathcal{A},S}(X)$;

$$(c) \quad \rho_{\text{int}(\mathcal{A}),S}(X) = \rho_{\mathcal{A},S}(X).$$

Proof. We only prove part (i) on lower semicontinuity. The proof of part (ii) proceeds along similar lines.

To prove that (a) and (b) are equivalent, note that by S -additivity $\rho_{\mathcal{A},S}$ is lower semicontinuous at X if and only if for any $m < \rho_{\mathcal{A},S}(X)$ there exists a neighborhood \mathcal{U} of X such that $\rho_{\mathcal{A},S}(Y + mS_1) > 0$ for all $Y \in \mathcal{U}$, i.e. $Y + mS_1 \notin \mathcal{A}$ for all $Y \in \mathcal{U}$. But this is equivalent to $X + mS_1$ lying outside $\overline{\mathcal{A}}$ for any $m < \rho_{\mathcal{A},S}(X)$. Hence (a) and (b) are indeed equivalent.

Now assume (b). Then $\rho_{\overline{\mathcal{A}},S}(X) \geq \rho_{\mathcal{A},S}(X)$. Since the opposite inequality is always satisfied, it follows that (c) holds. On the other hand, (b) follows immediately from (c), since $\rho_{\mathcal{A},S}(X) = \rho_{\overline{\mathcal{A}},S}(X) = \inf\{m \in \mathbb{R}; X + mS_1 \in \overline{\mathcal{A}}\}$. \square

As a consequence of the previous proposition, we obtain the following characterization of the lower and upper semicontinuity of $\rho_{\mathcal{A},S}$ on the full space \mathcal{X} .

Proposition 6.2 (Global semicontinuity). *Consider an acceptance set $\mathcal{A} \subset \mathcal{X}$ and a traded asset S .*

(i) *The following statements are equivalent:*

- (a) $\rho_{\mathcal{A},S}$ is (globally) lower semicontinuous;
- (b) $\{X \in \mathcal{X}; \rho_{\mathcal{A},S}(X) \leq 0\}$ is closed;
- (c) $\overline{\mathcal{A}} = \{X \in \mathcal{X}; \rho_{\mathcal{A},S}(X) \leq 0\}$.

(ii) *The following statements are equivalent:*

- (a) $\rho_{\mathcal{A},S}$ is (globally) upper semicontinuous;
- (b) $\{X \in \mathcal{X}; \rho_{\mathcal{A},S}(X) < 0\}$ is open;
- (c) $\text{int}(\mathcal{A}) = \{X \in \mathcal{X}; \rho_{\mathcal{A},S}(X) < 0\}$.

Proof. As in the previous proposition we only prove part (i). Clearly, (a) implies (b). Moreover, by Remark 2.10 we have $\mathcal{A} \subset \{X \in \mathcal{X}; \rho_{\mathcal{A},S}(X) \leq 0\} \subset \overline{\mathcal{A}}$. Therefore, (b) implies (c). Finally, if (c) holds then $\{X \in \mathcal{X}; \rho_{\mathcal{A},S}(X) \leq \lambda\} = \lambda S_1 + \overline{\mathcal{A}}$ for every $\lambda \in \mathbb{R}$, hence (a) follows. \square

Remark 6.3. Consider an acceptance set $\mathcal{A} \subset \mathcal{X}$ and a traded asset S .

- (i) As can be expected, if \mathcal{A} is closed, then $\rho_{\mathcal{A},S}$ is lower semicontinuous for any choice of S . If \mathcal{A} is open, then $\rho_{\mathcal{A},S}$ is upper semicontinuous for any choice of S . Both properties follow easily from the fundamental inclusions in Remark 2.10.
- (ii) If there exists $X \in \mathcal{X}$ such that $\rho_{\mathcal{A},S}(X) < \infty$ and $\rho_{\mathcal{A},S}$ is upper semicontinuous at X , then necessarily $\text{int}(\mathcal{A}) \neq \emptyset$. In particular if $\rho_{\mathcal{A},S}$ is upper semicontinuous on \mathcal{X} then $\text{int}(\mathcal{A}) \neq \emptyset$. This follows immediately from property (c), part (ii), of the previous proposition.

Remark 6.4. Let $\mathcal{A} \subset \mathcal{X}$ be an acceptance set and S a traded asset. Assume that \mathcal{A} is convex and $\rho_{\mathcal{A},S}$ is lower semicontinuous. If $\rho_{\mathcal{A},S}(X) = -\infty$ for some $X \in \mathcal{X}$, then $\text{dom}_{\mathbb{R}}(\rho_{\mathcal{A},S}) = \emptyset$, i.e. the risk measure $\rho_{\mathcal{A},S}$ cannot assume finite values. This is a consequence of a classical result on lower semicontinuous convex functions (see [13, Proposition 2.4] for a proof).

We close this appendix by showing that a capital requirement may be neither lower nor upper semicontinuous.

Example 6.5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a nonatomic probability space and define the set

$$\mathcal{A} := \{X \in L^1; \mathbb{E}[X] \geq 0 \text{ and } \exists \lambda \in \mathbb{R} : X \geq \lambda \text{ almost surely}\} . \quad (23)$$

It is easy to see that \mathcal{A} is a convex acceptance set which has empty interior. Furthermore, we have $\overline{\mathcal{A}} = \{X \in L^1; \mathbb{E}[X] \geq 0\}$, which is a closed convex acceptance set with nonempty interior. In particular, \mathcal{A} is a convex acceptance set for which the closure of its interior does not coincide with the interior of its closure. While $\rho_{\overline{\mathcal{A}}, S}$ is continuous for any choice of a nonzero $S_1 \in L^1_+$, the capital requirement $\rho_{\mathcal{A}, S}$ is neither upper semicontinuous — since $\text{int}(\mathcal{A}) = \emptyset$ — nor lower semicontinuous — since otherwise $\rho_{\mathcal{A}, S} = \rho_{\overline{\mathcal{A}}, S}$, and $\rho_{\overline{\mathcal{A}}, S}$ is continuous.

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WALTER FARKAS
University of Zurich and ETH Zürich
Department of Banking and Finance
Plattenstrasse 14
8032 Zürich, Switzerland
Email: walter.farkas@bf.uzh.ch

PABLO KOCH-MEDINA
Swiss Reinsurance Company
Mythenquai 50/60
8022 Zürich, Switzerland
Email: Pablo.KochMedina@swissre.com

COSIMO-ANDREA MUNARI
ETH Zürich
Department of Mathematics
Rämistrasse 101
8092 Zürich, Switzerland
Email: munari@math.ethz.ch